CHAPTER

8

Numerical Methods in Heat Conduction

8.1 Introduction

In chapter 3, we derived the general differential equation for heat conduction in cartesian, cylindrical and spherical coordinates. Subsequently, considering one-dimensional conduction, we solved these differential equations, with appropriate boundary conditions, for cases of simple geometries such as a plane wall, cylinder and sphere and obtained temperature distribution in those geometries; then, by applying Fourier's law, heat transfer rate was obtained. The analytical solutions obtained for temperature distribution are known as 'exact solutions' since temperature at any point in the solid is obtained by applying the equations derived. While getting an exact solution is always preferable, following points in connection with the analytical solutions must be noted:

(i) Analytical solutions are suitable for simple geometries such as a plane wall, cylinder or sphere, where the surface of the body and the coordinate surfaces coincide, i.e. surfaces of a plane wall are completely bounded by the coordinate surfaces of a cartesian coordinate system, surfaces of a cylinder and sphere are completely bounded by a cylindrical and spherical coordinate system respectively.

(ii) However, for irregular geometries, analytical solutions become difficult. For example, if there is a handle on a cylindrical cup, finding out the temperature distribution in the system becomes very difficult or impossible by analytical methods.

(iii) Further, even in simple geometries, if there is variation of thermal conductivity with temperature, or if the heat transfer coefficient varies over the surface, or if there is radiation heat transfer involved at the surfaces, severe non-linearities are introduced and analytical solutions become highly complicated or impossible.

(iv) Many times, analytical solutions, even if available for certain problems, are so complicated with the presence of infinite series, Bessel functions etc. that the user gets intimidated from using them.

In such cases, popular alternative method is 'numerical solution'. Here, the differential equation is substituted by a set of algebraic equations and simultaneous solution of these algebraic equations gives the temperatures at selected, 'discrete points' in the system. So, the important difference to be noted is that while in an analytical solution, temperature is obtained at any point in the body, in a numerical solution temperatures are obtained only at selected, discrete points or 'nodes'. By selecting these nodes close enough, sufficiently accurate results are obtained.

Advantages of numerical methods are:

(i) easy to apply, with the availability of high speed computers

(ii) desired accuracy can be obtained by controlling the number of nodes or 'mesh size'.

(iii) variation in area, thermal conductivity or heat transfer coefficients, and complicated boundary conditions can be easily taken into account.

(iv) mathematical model for a numerical solution is more likely to be a better representative of the actual system

(v) parametric study to observe the effect of variation of different parameters on the solution, or 'what-if' analysis, is easier with numerical methods in conjunction with high speed computers.

Generally used numerical techniques are 'finite difference', 'finite element', 'boundary element' and 'energy balance or control volume' methods. We will adopt energy balance method since it is intuitively easier to apply energy balance on control volumes and does not involve complicated mathematical formulations.

In this chapter, we shall learn to formulate set of algebraic equations from the differential equations in cartesian, cylindrical and spherical coordinates and solve them for one-dimensional, steady state conduction. Then, we shall study the finite difference representation and solution of two-dimensional, steady state conduction problems. Since the numerical solution essentially involves solving a set of algebraic equations simultaneously, we shall study the different methods of solving simultaneous algebraic equations. Finally, numerical solution of one-dimensional and two-dimensional transient conduction problems will be described.

8.2 Finite Difference Formulation from Differential Equations

As mentioned earlier, in this book, we shall formulate finite difference equations by making energy balance on differential control volumes. However, as an introduction and as an example, for one case, let us obtain the finite difference form of equation directly from the differential equation mathematically, starting with the definition of first derivative and second derivatives.

Consider the governing equation for one-dimensional, steady state heat conduction with heat generation:

$$\frac{d^2T(x)}{dx^2} + \frac{q_g}{k} = 0 \text{ in } 0 < x < L \qquad ...(8.1)$$

Now, le us divide the region $0 < x < L$ into M sub-regions. Then, size of each sub-region is:

$$\Delta x = \frac{L}{M} \qquad ...(8.2)$$

So, there are $M + 1$ nodes, starting from $m = 0$ to $m = M$, as shown in Fig. 8.1.

Coordinate of node m is $x = m.\Delta x$. and let temperature of node m be T_m.

Now, in Eq. 8.1, we need second derivative of T. To represent it in terms of finite differences, we proceed as follows:

Consider locations (m + ½) and (m – ½) as shown in Fig. 8.1. First derivative of temperature dT/dx at these locations is written in terms of finite differences as:

$$\left(\frac{dT(x)}{dx}\right)_{m+\frac{1}{2}} = \frac{T_{m+1} - T_m}{\Delta x} \qquad ...(8.3a)$$

and,

$$\left(\frac{dT(x)}{dx}\right)_{m-\frac{1}{2}} = \frac{T_m - T_{m-1}}{\Delta x} \qquad ...(8.3b)$$

Then, the second derivative d^2T/dx^2 at node m is approximated as:

$$\left(\frac{d^2T(x)}{dx^2}\right)_m = \frac{\left(\frac{dT(x)}{dx}\right)_{m+\frac{1}{2}} - \left(\frac{dT(x)}{dx}\right)_{m-\frac{1}{2}}}{\Delta x}$$

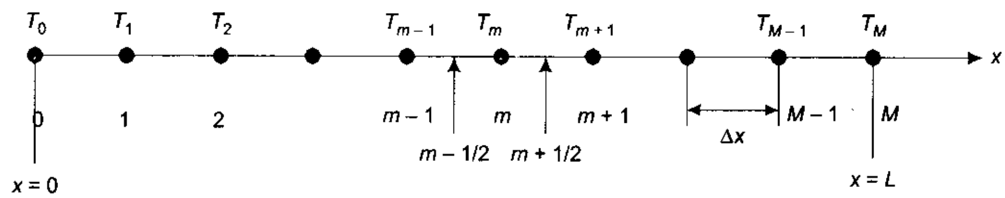


FIGURE 8.1 Finite difference representation of derivatives

i.e. $$\left(\frac{d^2T(x)}{dx^2}\right)_m = \frac{T_{m-1} - 2\cdot T_m + T_{m+1}}{(\Delta x)^2} \quad ...(8.4)$$

Substituting Eq. 8.4 in Eq. 8.1:

$$(T_{m-1} - 2\cdot T_m + T_{m+1}) + \frac{(\Delta x)^2 \cdot q_m}{k} = 0 \quad ...(8.5)$$

where, q_m is the energy generation rate at node m, and $m = 1, 2, 3.....M - 1$

Eq. 8.5 is the finite difference form of representation of the differential equation given by Eq. 8.1. It is valid for the 'interior nodes' i.e. nodes 1, 2....$M - 1$. Since q_m, k and Δx are known quantities, Eq. 8.5 provides $(m - 1)$ simultaneous algebraic equations for temperature. But, there are $M + 1$ nodes, and we need two more equations to solve $M + 1$ node temperatures; these two equations are obtained by finite difference representation of boundary conditions at nodes $m = 0$ and $m = M$, as will be shown later.

8.3 One-dimensional, Steady State Heat Conduction in Cartesian Coordinates

Now, we shall develop the finite difference formulation using the 'energy balance' approach. In this method, the medium in question is sub-divided into many sub-volumes; centre of each sub-volume is known as a 'node' and each node represents the average properties of the sub-volume around it. Thus, at node 'm', temperature T_m of that node represents the average temperature of the sub-volume around node 'm'. It is imagined that these nodes are connected to each other by 'conducting rods' i.e. in effect, the total volume is replaced by a network of nodes with conducting rods. It is further assumed that temperature between adjacent nodes varies linearly.

Consider one-dimensional, steady state heat conduction in a plane wall of thickness L, with heat generation rate $q_g(x)$ and constant thermal conductivity k. Now, let us divide the region $0 < x < L$ into M sub-regions. Then, thickness of each sub-region is:

$\Delta x = L/M$. So, there are totally $(M + 1)$ nodes, starting from $m = 0$ to $m = M$, as shown in Fig. 8.2. Coordinate of node m is $x = m.\Delta x$. and let temperature of node m be T_m. Remembering that each node represents the sub-volume around it (of thickness Δx), it is clear that interior nodes 1, 2...$M - 1$ represent full sub-volumes whereas boundary nodes 0 and M represent half volumes (of thickness $\Delta x/2$).

To get the difference equation for the interior nodes, let us write an energy balance for the volume element represented by node m. Assuming that all heat conduction is ***into*** the element, we can write, for steady state conditions:

Rate of heat conduction from left + Rate of heat conduction from right + Rate of heat generation inside the element = 0. i.e.

$$Q_{left} + Q_{right} + q_m \cdot A \cdot \Delta x = 0 \quad ...(8.6)$$

where q_m is the heat generation rate per unit volume for sub-volume represented by node m (assumed constant for the entire wall), A is the heat transfer area perpendicular to the direction of heat flow (constant for the wall), and A. Δx is the volume of the element. Now, note that for a wall with heat generation, temperature distribution is *not* linear. However, we make an approximation that the temperature variation between two nodes is linear; and this assumption is valid for small values of Δx. So, writing the energy balance, with the direction of all heat flow ***into*** the element,

$$k\cdot A\cdot\frac{T_{m-1} - T_m}{\Delta x} + k\cdot A\cdot\frac{T_{m+1} - T_m}{\Delta x} + q_m\cdot A\cdot\Delta x = 0 \quad ...(8.7)$$

i.e. $$(T_{m-1} - 2\cdot T_m + T_{m+1}) + \frac{q_m\cdot(\Delta x)^2}{k} = 0 \quad ...(8.8)$$

where, q_m is the energy generation rate at node m, and $m = 1,2,3.....M - 1$

Note that Eq. 8.8 is identical to Eq. 8.5 derived earlier mathematically by consideration of definition of first and second derivatives.

Again, Eq. 8.8 is applicable only to $M - 1$ interior nodes; we will need two more equations to solve M unknown node temperatures. These two equations are obtained by writing energy balance at the two boundary nodes 0 and M.

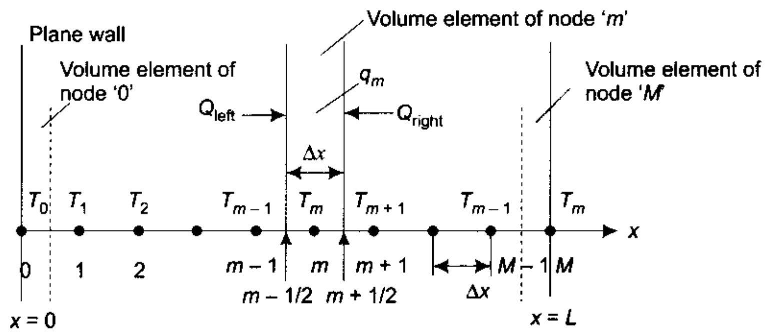


FIGURE 8.2 Finite difference formulation in a plane wall by energy balance

It is convenient to assume while writing the energy balance, that all heat flows are *towards* the node in question; the signs of heat flow adjust themselves when the set of coupled algebraic equations so obtained are solved simultaneously.

Writing in terms of thermal resistances, Eq. 8.7 can be written as:

$$\frac{T_{m-1} - T_m}{R_{m-1,m}} + \frac{T_{m+1} - T_m}{R_{m+1,m}} + q_m \cdot A \cdot \Delta x = 0 \quad ...(8.9)$$

where $R_{m-1,m} = \left(\frac{\Delta x}{k \cdot A}\right)_{m-1,m}$ = thermal resistance between nodes $m - 1$ *and* m

$R_{m+1,m} = \left(\frac{\Delta x}{k \cdot A}\right)_{m+1,m}$ = thermal resistance between nodes $m + 1$ and m

q_m = energy generation rate at node m

$A \cdot \Delta x$ = volume of element about node m

Eq. 8.9 is more general and allows for the variation of thermal conductivity and cross-sectional area with position. When k and A are constants, Eq. (8.9) reduces to eqn. (8.8).

Boundary conditions:

Eq. 8.8 or 8.9 developed above are applicable to internal nodes. For nodes at the boundaries (i.e. for nodes 0 and M), difference equations are developed again by writing the energy balance for the volume elements containing these nodes. While doing so, the boundary conditions prescribed in the problem must be taken into account. Also, note that volume elements for nodes '0' and 'M' for a plane wall are half-volumes as shown in Fig. 8.2.

Most commonly encountered boundary conditions are: prescribed temperature, prescribed heat flux, convection and radiation boundary conditions.

(i) Prescribed temperatures at the boundaries This is the simplest of the boundary conditions. Let the temperatures at $x = 0$ and $x = L$ be given as T_o and T_M respectively. Then, $T(o) = T_o$ and $T(L) = T_M$, give the two additional equations required to solve for $M + 1$ unknown node temperatures. In this case, there is no need to write energy balance for volume elements at the boundaries, since the temperatures at the boundaries are known.

To develop finite difference equations for the other boundary conditions, we apply energy balance to the volume elements of nodes at the boundaries, i.e. nodes 0 and M (See Fig. 8.3). Remember that these volume elements are only half volumes of thickness $\Delta x/2$, each. Also, while writing energy balance, consider all energy flows as flowing ***into*** the element. Heat flux into the element is considered as positive and out of the element, it is negative.

Then, energy balance for the volume element for node '0' on the left boundary of the wall is given by:

$$Q_{left} + k \cdot A \cdot \frac{(T_1 - T_0)}{\Delta x} + q_o \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \quad ...(8.10)$$

Let us now apply Eq. 8.10 to get difference equations for boundary nodes '0' and 'M':

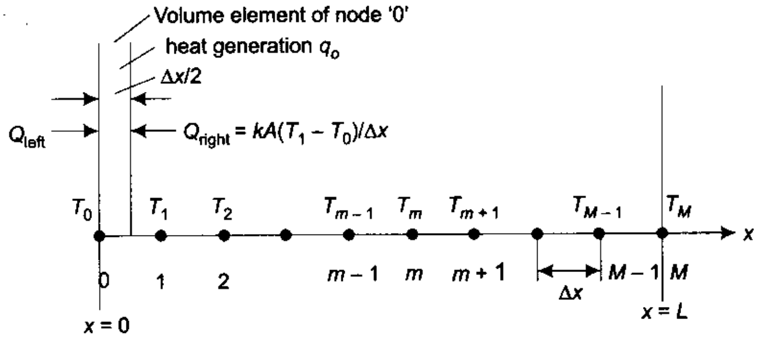


FIGURE 8.3 Finite difference formulation for left boundary of a plane wall

(ii) Prescribed heat flux at the boundaries Let q_{left} and q_{right} be the heat flux at nodes '0' and 'M' respectively. Then, from Eq. 8.10:

For node '0':

$$q_{left} \cdot A + k \cdot A \cdot \frac{(T_1 - T_0)}{\Delta x} + q_o \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad ...(8.11)$$

i.e.

$$2 \cdot T_1 - 2 \cdot T_0 + \frac{(\Delta x)^2 \cdot q_o}{k} + \frac{2 \cdot \Delta x \cdot q_{left}}{k} = 0 \qquad ...(8.12)$$

For node 'M': Replace the subscript '0' by 'M' and subscript '1' by '$M - 1$':

$$q_{right} \cdot A + k \cdot A \cdot \frac{(T_{M-1} - T_M)}{\Delta x} + q_M \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad ...(8.13)$$

i.e.

$$2 \cdot T_{M-1} - 2 \cdot T_M + \frac{(\Delta x)^2 \cdot q_M}{k} + \frac{2 \cdot \Delta x \cdot q_{right}}{k} = 0 \qquad ...(8.14)$$

Eqs. 8.12 and 8.14 are finite difference representation of the prescribed heat flux conditions at nodes '0' and 'M' respectively.

For insulated boundary condition and for a plane of thermal symmetry:

This is a special case of prescribed heat flux condition. Now, $q_{left} = q_{right} = 0$. Then, Eqs. 8.12 and 8.14 become:

$$2 \cdot T_1 - 2 \cdot T_0 + \frac{(\Delta x)^2 \cdot q_o}{k} = 0 \qquad ...(8.15)$$

$$2 \cdot T_{M-1} - 2 \cdot T_M + \frac{(\Delta x)^2 \cdot q_M}{k} = 0 \qquad ...(8.16)$$

Eq. 8.15 and 8.16 for an insulated boundary or a plane of thermal symmetry can be obtained more easily by applying the **'mirror image concept'**. In this simple method, the insulated boundary or the plane of thermal symmetry is considered as a mirror. Thus, for the node '0', insulated left face becomes a mirror and reflects node 1; then, node '0' has the reflected node '1' on its left and node '1' on its right and we write the difference equation as if the node '0' is an internal node. Then, applying Eq. 8.8 for an internal node, putting $m = 0$, we get:

$$(T_{m-1} - 2 \cdot T_m + T_{m+1}) + \frac{q_m \cdot (\Delta x)^2}{k} = 0 \qquad ...(8.8)$$

Put $m = 0$ and $T_{-1} = T_1$:

i.e.

$$2 \cdot T_1 - 2 \cdot T_0 + \frac{q_o \cdot (\Delta x)^2}{k} = 0 \qquad ...(8.17)$$

Equation (8.17) is the same as eqn. (8.15).

Similarly, for node 'M', right hand surface which is insulated becomes the mirror and node '$M - 1$' is reflected further to the right of node 'M' and now, considering node 'M' as an internal node, Eq. 8.8 becomes:

$$(T_{m-1} - 2 \cdot T_m + T_{m+1}) + \frac{q_m \cdot (\Delta x)^2}{k} = 0 \qquad \text{...(8.8)}$$

Put $m = M$, and $M + 1 = M - 1$:

i.e.
$$2 \cdot T_{M-1} - 2 \cdot T_M + \frac{(\Delta x)^2 \cdot q_M}{k} = 0 \qquad \text{...(8.18)}$$

Eq. 8.18 is the same as Eq. 8.16.

Thus, note that for an insulated boundary condition, or for a plane of thermal symmetry, it is very convenient to use the 'mirror image concept' and write the difference equation as if the boundary node is an internal node.

(iii) Convection boundary condition Let the boundaries at $x = 0$ and $x = L$ be subjected to convection to a fluid at a temperature of T_a with a heat transfer coefficient of h.

Then, Eq. 8.10 becomes:

For node '0':

$$h \cdot A \cdot (T_a - T_0) + k \cdot A \cdot \frac{(T_1 - T_0)}{\Delta x} + q_0 \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad \text{...(8.19)}$$

i.e.
$$2 \cdot T_1 - 2 \cdot T_0 \cdot \left(1 + \frac{h \cdot \Delta x}{k}\right) + \frac{(\Delta x)^2 \cdot q_o}{k} + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a = 0 \qquad \text{...(8.20)}$$

For node 'M': Replace the subscript '0' by 'M' and subscript '1' by '$M - 1$':

We get:

$$h \cdot A \cdot (T_a - T_M) + k \cdot A \cdot \frac{(T_{M-1} - T_M)}{\Delta x} + q_M \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad \text{...(8.21)}$$

i.e.
$$2 \cdot T_{M-1} - 2 \cdot T_M \cdot \left(1 + \frac{h \cdot \Delta x}{k}\right) + \frac{(\Delta x)^2 \cdot q_M}{k} + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a = 0 \qquad \text{...(8.22)}$$

Eq. 8.20 and 8.22 are finite difference representations for convective boundary conditions at nodes '0' and 'M' respectively.

(iv) Radiation boundary condition Let the surrounding temperature be T_a, emissivity of the surface ε, and σ, the Stefan– Boltzmann constant. Then, Eq. 8.10 becomes:

For node '0':

$$\varepsilon \cdot \sigma \cdot A \cdot (T_a^4 - T_0^4) + k \cdot A \cdot \frac{(T_1 - T_0)}{\Delta x} + q_0 \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad \text{...(8.23)}$$

For node 'M': Replace the subscript '0' by 'M' and subscript '1' by '$M - 1$':

We get:

$$\varepsilon \cdot \sigma \cdot A \cdot (T_a^4 - T_M^4) + k \cdot A \cdot \frac{T_{M-1} - T_M}{\Delta x} + q_M \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad \text{...(8.24)}$$

We generally try to avoid radiation boundary condition even with numerical methods, since as can be seen easily from Eqs. 8.23 and 8.24, finite difference equations now become highly non-linear and are difficult to solve.

(v) Combined convection and radiation boundary condition Let there be radiation as well as convection at the surfaces, giving a combined heat transfer coefficient of h_{comb} and let the fluid temperature be T_a. Then, Eq. 8.10 becomes:

For node '0':

$$h_{\text{comb}} \cdot A \cdot (T_a - T_0) + k \cdot A \cdot \frac{(T_1 - T_0)}{\Delta x} + q_0 \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \qquad \text{...(8.25)}$$

i.e. $$2 \cdot T_1 - 2 \cdot T_0 \cdot \left(1 + \frac{h_{comb} \cdot \Delta x}{k}\right) + \frac{(\Delta x)^2 \cdot q_0}{k} + \frac{2 \cdot h_{comb} \cdot \Delta x}{k} \cdot T_a = 0 \quad ...(8.26)$$

For node 'M': Replace the subscript '0' by 'M' and subscript '1' by 'M – 1':
We get:

$$h_{comb} \cdot A \cdot (T_a - T_M) + k \cdot A \cdot \frac{(T_{M-1} - T_M)}{\Delta x} + q_M \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \quad ...(8.27)$$

i.e. $$2 \cdot T_{M-1} - 2 \cdot T_M \cdot \left(1 + \frac{h_{comb} \cdot \Delta x}{k}\right) + \frac{(\Delta x)^2 \cdot q_M}{k} + \frac{2 \cdot h_{comb} \cdot \Delta x}{k} \cdot T_a = 0 \quad ...(8.28)$$

(vi) Interface boundary condition With no contact resistance:
Node 'm' is at the interface between two solids in 'perfect thermal contact', i.e. there is no contact resistance and both the surfaces are at the same temperature at the interface node 'm'. This situation is shown in Fig. 8.4.

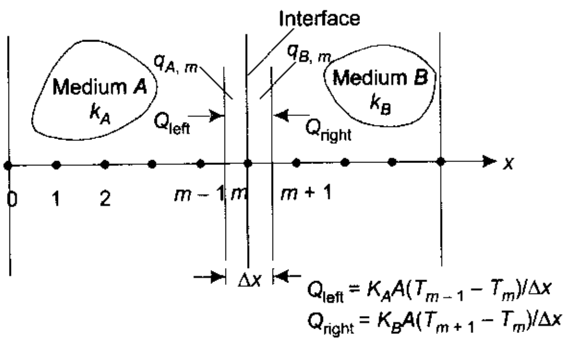


FIGURE 8.4 Finite difference formulation for interface boundary condition

So, the finite difference formulation for this boundary condition is given by:

$$k_A \cdot A \cdot \frac{T_{m-1} - T_m}{\Delta x} + k_B \cdot A \cdot \frac{T_{m+1} - T_m}{\Delta x} + q_{A,m} \cdot \left(\frac{A \cdot \Delta x}{2}\right) + q_{B,m} \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \quad ...(8.29)$$

In the above relation, subscripts A and B refer to materials A and B, k is the thermal conductivity, q is the heat generation rate and A is the area of cross-section normal to the direction of heat flow.
With contact resistance:
If there is a contact resistance R_c at the interface, we use the resistance concept to write the difference equation. (See Eq. 8.9). Now, at the interface, there is a temperature drop. Let the temperature at the interface drop from Tc_1 to Tc_2.

Then, we can write:

$$k_A \cdot A \cdot \frac{T_{m-1} - Tc_1}{\Delta x} + k_B \cdot A \cdot \frac{T_{m+1} - Tc_2}{\Delta x} + q_{A,m} \cdot \left(\frac{A \cdot \Delta x}{2}\right) + q_{B,m} \cdot \left(\frac{A \cdot \Delta x}{2}\right) = 0 \quad ...(8.30)$$

And, temperature drop at the interface is calculated as:

$$\Delta T_c = (Tc_1 - Tc_2) = Q \cdot \frac{R_c}{A} \quad ...(8.31)$$

where Q is the heat flow rate through the interface (i.e. between nodes (m – 1) and (m + 1)) and (R_c/A) is the interface thermal resistance.

Example 8.1. One face of a slab of thickness 1 cm (k = 20 W/(mC)), is maintained at 40°C and the other surface is subjected to a convection heat transfer with a fluid at 100°C with a heat transfer coefficient of 4000 W/(m^{2}C). There is uniform internal heat generation in the slab at a rate of 8×10^7 W/m^3.

(a) Dividing the slab into 5 equally spaced sub-regions, find the temperatures at the different nodes. Assume one-dimensional, steady state conduction.

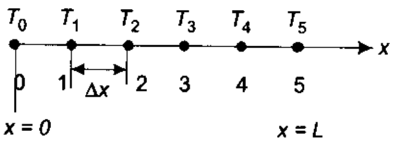


FIGURE Example 8.1 Finite difference nodes for Example 8.1

(b) If the left surface is insulated, what is the temperature on that surface in steady state?

Solution.

Data:

$L := 0.01$ m $\quad M := 5 \quad k := 20$ W/(mC) $\quad T_0 := 40°C$

$T_a := 100°C \quad h := 4000$ W/(mC) $\quad q_g := 8 \times 10^7$ W/m^3

$$\Delta x := \frac{L}{M} = \frac{0.01}{5} \quad \text{i.e. } \Delta x = 0.002 \text{ m}$$

Note that there are 6 nodes, numbered as: 0, 1, 2, 3, 4, and 5. Out of these, nodes '0' and '5' are boundary nodes and the nodes 1, 2, 3 and 4 are internal nodes. Temperature of node '0' is given, i.e. $T_0 = 40°C$, for the first part of the problem.

Fig. Example 8.1 shows the schematic of finite difference nodes for this problem.

Apply Eq. 8.8 for interior nodes, 1, 2, 3 and 4:

$$(T_{m-1} - 2 \cdot T_m + T_{m+1}) + \frac{q_g \cdot (\Delta x)^2}{k} = 0 \qquad (8.8)$$

We have: $$\frac{q_g \cdot (\Delta x)^2}{k} = 16$$

Node 0: $T_0 = 40°C$ *(by data...(a))*

Node 1: $T_0 - 2 \cdot T_1 + T_2 + 16 = 0$ ...(b)

Node 2: $T_1 - 2 \cdot T_2 + T_3 + 16 = 0$ ...(c)

Node 3: $T_2 - 2 \cdot T_3 + T_4 + 16 = 0$ ...(d)

Node 4: $T_3 - 2 \cdot T_4 + T_5 + 16 = 0$ ...(e)

For Node 5: here, we have convection boundary condition. So, apply Eq. 8.22:

$$2 \cdot T_{M-1} - 2 \cdot T_M \cdot \left(1 + \frac{h \cdot \Delta x}{k}\right) + \frac{(\Delta x)^2 \cdot q_M}{k} + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a = 0 \qquad ...(8.22)$$

Here, $q_M = q_g$

Then, for $M = 5$, we get:

$$2 \cdot T_4 - 2.8 \cdot T_5 + 16 + 80 = 0 \qquad (f)$$

Eq. a to f have to be solved simultaneously to get 6 nodal temperatures. Of course, in this case temperature at node '0' is already known.

We shall discuss the different methods of solving coupled algebraic equations, later. But, now, we will use 'Solve block' of Mathcad to solve these 6 equations simultaneously.

We start with assumed or trial values for all the variables i.e. for the temperatures at nodes 1 to 5. Then, in the solve block, immediately below 'Given' write all the constraint equations. Then, the command 'Find (T_0, T_1, T_2 ... T_5)' immediately gives a vector of temperature values:

T_1: = 50 $\quad T_2$: = 50 $\quad T_3$: = 50 $\quad T_4$: = 50 $\quad T_5$: = 50 *(trial values of temperatures)*

Given

$T_0 = 40°C$ *(by data...(a))*

$T_0 - 2 \cdot T_1 + T_2 + 16 = 0$ ...(b)

$T_1 - 2 \cdot T_2 + T_3 + 16 = 0$ ...(c)

$T_2 - 2 \cdot T_3 + T_4 + 16 = 0$ ...(d)

$T_3 - 2 \cdot T_4 + T_5 + 16 = 0$ ...(e)

$2 \cdot T_4 - 2 \cdot 8 \cdot T_5 + 16 + 80 = 0$ ...(f)

Temp: = Find(T_0, T_1, T_2, T_3, T_4, T_5) *('Temp' is the vector containing values of temperatures T_0, T_1 ... T_5)*

Therefore,

$$\text{Temp} = \begin{bmatrix} 40 \\ 93.333 \\ 130.667 \\ 152 \\ 157.333 \\ 146.667 \end{bmatrix}$$

i.e. Temperatures at different nodes are:

$T_0 = 40°C$ $T_1 = 93.333°C$ $T_2 = 130.667°C$ $T_3 = 152°C$ $T_4 = 157.333°C$ $T_5 = 146.667°C$

To draw the temperature distribution:

In the above, temperatures at various nodes are contained in vector 'Temp'. In Mathcad, elements of the vector are generally counted starting from zero. i.e. zeroth element of vector Temp gives value of $T_0 = 40$, element numbered 1 gives value of $T_1 = 93.333$, and so on.

To draw the graph, first we define a range variable $i = 0$ to 5. Then choose x–y graph from the graph pallete and on the x-axis place holder fill up i and in the y-axis place holder fill up $Temp_i$. Click anywhere outside the graph and immediately the graph appears.

$i := 0, 1, ..., 5$ *(define the range variable i, varying from* 0 *to* 5 *with an increment of* 1)

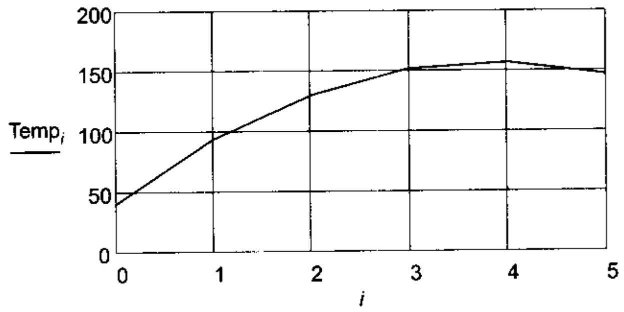


FIGURE Example 8.1(b) Temperature at different nodes in the slab

In the Fig. Example 8.1 (b) 'i' is the node no. on the x-axis and on the y-axis, $Temp_i$, the corresponding node temperature is plotted.

(b) When left surface is insulated:

Now, the node '0' is on an insulated boundary. Difference equation for node '0' is obtained now, treating it as an internal node if the insulated surface is imagined to be a mirror i.e. node '1' extends to the left of node '0' and Eq. 8.8 is applicable.

i.e. $$(T_{m-1} - 2\cdot T_m + T_{m+1} + \frac{q_m\cdot(\Delta x)^2}{k} = 0 \qquad ...(8.8)$$

For $m = 0$: $$T_{-1} - 2\cdot T_o + T_1 + \frac{q_g\cdot(\Delta x)^2}{k} = 0$$

From mirror image concept: $T_{-1} = T_1$

Therefore, for node '0', we get:

$$2\cdot T_1 - 2\cdot T_0 + 16 = 0 \qquad ...(a')$$

Equations for other nodes remain unchanged.

Therefore, solving Eq. a' alongwith b, c, d, e and f simultaneously will give the temperatures at nodes 0 to 5.

Use 'solve block' to solve the set of algebraic Eqs. a' to f simultaneously, in Mathcad. Start with assumed or trial values of temperatures:

$T_0 := 50$ $T_1 := 50$ $T_2 := 50$ $T_3 := 50$ $T_4 := 50$ $T_5 := 50$ *(trial values of temperatures)*

Given

$$2\cdot T_1 - 2\cdot T_0 + 16 = 0 \qquad ...(a')$$
$$T_0 - 2\cdot T_1 + T_2 + 16 = 0 \qquad ...(b)$$
$$T_1 - 2\cdot T_2 + T_3 + 16 = 0 \qquad ...(c)$$
$$T_2 - 2\cdot T_3 + T_4 + 16 = 0 \qquad ...(d)$$
$$T_3 - 2\cdot T_4 + T_5 + 16 = 0 \qquad ...(e)$$
$$2\cdot T_4 - 2.8\cdot T_5 + 16 + 80 = 0 \qquad ...(f)$$

Temp := Find(T_0, T_1, T_2, T_3, T_4, T_5) *('Temp' is the vector containing values of temperatures* T_0, T_1 ... T_5)

Therefore,

$$\text{Temp} = \begin{bmatrix} 500 \\ 492 \\ 468 \\ 428 \\ 372 \\ 300 \end{bmatrix}$$

i.e. Temperatures at different nodes are:

$T_0 = 500°C$ $T_1 = 492°C$ $T_2 = 468°C$ $T_3 = 428°C$ $T_4 = 372°C$ $T_5 = 300°C$

Let us compare these values with those obtained from analytical solution. Analytical solution to this problem is easily obtained from the following mathematical formulation of the problem:

$$\frac{d^2T(x)}{dx^2} + \frac{q_g(x)}{k} = \text{in } 0 < x < L \qquad \textit{(eqn. (h)...for a slab)}$$

$$\frac{dT(x)}{dx} = 0 \quad \text{at } x = 0 \qquad \textit{((i)...since insulated)}$$

$$h \cdot A \cdot (T_5 - T_a) = -k \cdot A \cdot \left(\frac{dT(x)}{dx}\right)_{x=L} \qquad \textit{...(j)...convection at the right surface, i.e. at } x = L$$

Solving the above governing differential Eq. h with the boundary conditions *i* and *j* at $x = 0$ and $x = L$, we get the following analytical solution for temperature distribution:

$$T(x) := \frac{q_g \cdot L^2}{2 \cdot k} \cdot \left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{q_g \cdot L}{h} + T_a \qquad \text{...(k)}$$

Then, temperatures at nodes '0' to 5 are obtained by putting corresponding x values in $T(x)$:

	Temperatures from Analytical solution	Temperatures from numerical soluion
Node 0:	$T(0) = 500°C$	$T_0 = 500°C$
Node 1:	$T(0.002) = 492°C$	$T_1 = 492°C$
Node 2:	$T(0.004) = 468°C$	$T_2 = 468°C$
Node 3:	$T(0.006) = 428°C$	$T_3 = 428°C$
Node 4:	$T(0.008) = 372°C$	$T_4 = 372°C$
Node 5:	$T(0.01) = 300°C$	$T_5 = 300°C$

So, we see that values of temperatures obtained by numerical methods match extremely well with the values obtained by 'exact' analytical solution, i.e. even with only 5 equal divisions of the slab, we get very accurate solution by numerical method. Hence its popularity.

Example 8.2. Consider a slab of thickness, $L = 1$ cm. Thermal conductivity of the slab material varies linearly with temperature as: $k(T) = 26.679\ (1 + 8.621 \times 10^{-4}\ T)$, W/(mC), where T is in deg. C. Surface at $x = 0$ is insulated and the other surface at $x = L$ is subjected to a convection heat transfer with a fluid at 100°C with a heat transfer coefficient of 4000 W/(m^{2}C).There is uniform internal heat generation in the slab at a rate of 8×10^7 W/m^3. Dividing the slab into 5 equally spaced sub-regions, find the temperatures at the different nodes. Assume one-dimensional, steady state conduction.

Solution.

Data.

$L := 0.01$ m $\quad M := 5 \quad k(T) = 26.679(1 + 8.621 \times 10^{-4} \cdot T)$ W/(mC)

This is of the form:$k(T) := k_0(1 + \beta \cdot T)$

where,$k_0 := 26.679$ W/(mC) and, $\beta := 8.621 \times 10^{-4}$ 1/C $\quad T_a := 100°C \quad h := 4000$ W/(m^{2}C) $\quad q_g := 8 \times 10^7$ W/m^3

$$\Delta x = \frac{L}{M} = \frac{0.01}{5} \quad \text{i.e. } \Delta x := 0.002 \text{ m}$$

Note that there are 6 nodes, numbered as: 0, 1, 2, 3, 4, and 5. Out of these, nodes '0' and '5' are boundary nodes and the nodes 1, 2, 3 and 4 are internal nodes.

Fig. Example 8.2 shows the schematic of finite difference nodes for this problem.

Now, for the interior nodes, Eq. 8.8 is not applicable since the thermal conductivity varies with temperature.

Let us derive the difference equation for the interior nodes first. Consider any internal node 'm' and apply the energy balance for the differential volume around node 'm'. Remember to consider that all energy flows are into the control volume. Using the thermal resistance concept, we get:

$$\frac{T_{m-1}-T_m}{\dfrac{\Delta x}{k_0\cdot\left[1+\beta\cdot\left(\dfrac{T_{m-1}+T_m}{2}\right)\right]\cdot A}}+\frac{T_{m+1}-T_m}{\dfrac{\Delta x}{k_0\cdot\left[1+\beta\cdot\left(\dfrac{T_{m+1}+T_m}{2}\right)\right]\cdot A}}+q_m\cdot(A\cdot\Delta x)=0$$

i.e. $$(T_{m-1}-T_m)\cdot k_0\cdot\left[1+\beta\cdot\left(\frac{T_{m-1}+T_m}{2}\right)\right]+(T_{m+1}-T_m)\cdot k_0\cdot\left[1+\beta\cdot\left(\frac{T_{m+1}+T_m}{2}\right)\right]+q_m\cdot(\Delta x)^2=0$$

i.e. $$(T_{m-1}-2\cdot T_m+T_{m+1})+\frac{\beta}{2}\cdot[(T_{m-1})^2-2\cdot(T_m)^2+(T_{m+1})^2]+\frac{q_m\cdot(\Delta x)^2}{k_0}=0 \qquad \text{...(A)}$$

Eq. A gives the difference equation for the interior nodes 1, 2, 3, and 4. It is seen that this equation is non-linear and solving the set of non-linear equations by conventional methods is difficult. But, as we shall presently see, in Mathcad, it is very easy to get solution using the 'solve block'.

We have: $q_m = q_g$ and, $\dfrac{q_g\cdot(\Delta x)^2}{k_0}=11.994$

In Eq. A, let us put m =1, 2, 3 and 4 to get the difference equations for the respective nodes:

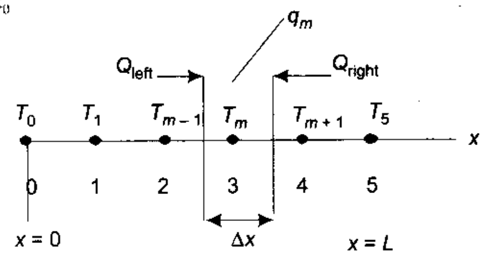


FIGURE Example 8.2 Finite difference nodes for Example 8.2

Node 1: $(T_0-2\cdot T_1+T_2)+\frac{\beta}{2}\cdot[(T_0)^2-2\cdot(T_1)^2+T_2)^2]+11.994=0$ ...(b)

Node 2: $(T_1-2\cdot T_2+T_3)+\frac{\beta}{2}\cdot[(T_1)^2-2\cdot(T_2)^2+T_3)^2]+11.994=0$ ...(c)

Node 3: $(T_2-2\cdot T_3+T_4)+\frac{\beta}{2}\cdot[(T_2)^2-2\cdot(T_3)^2+T_4)^2]+11.994=0$ ...(d)

Node 4: $(T_3-2\cdot T_4+T_5)+\frac{\beta}{2}\cdot[(T_3)^2-2\cdot(T_4)^2+T_5)^2]+11.994=0$ ...(e)

Difference equations for boundary nodes:

For node '0': Apply the energy balance to the half-volume around the node '0'; all heat lines flowing into the volume.

There is no heat flowing from the left side of the control volume into node '0' since the surface is insulated. Writing other terms, we get:

$$\frac{T_1-T_0}{\dfrac{\Delta x}{\left[k_0\cdot\left[1+\beta\cdot\left(\dfrac{T_0+T_1}{2}\right)\right]\cdot A\right]}}+q_g\cdot\left(A\cdot\frac{\Delta x}{2}\right)=0$$

i.e. $$(T_1-T_0)\cdot k_0\cdot\left[1+\beta\cdot\left(\frac{T_0+T_1}{2}\right)\right]+\frac{q_g\cdot(\Delta x)^2}{2}=0$$

i.e. $$(T_1-T_0)+\frac{\beta}{2}\cdot(T_1^2-T_0^2)+\frac{q_g\cdot(\Delta x)^2}{2\cdot k_0}=0$$

i.e. $$(T_1-T_0)+\frac{\beta}{2}\cdot(T_1^2-T_0^2)+5.997=0 \qquad \text{...(a)}$$

Eq. a is the difference equation for node '0'. This equation is also a non-linear equation

For node 5: Apply the energy balance to the half-volume around the node 5; all heat lines flowing into the volume. There is convection condition on the right surface. Writing the energy balance, we get:

$$\frac{\frac{T_4 - T_5}{\Delta x}}{\left[k_0 \cdot \left[1 + \beta \cdot \left(\frac{T_4 + T_5}{2}\right)\right] \cdot A\right]} + h \cdot A\,(T_a - T_5) + q_g \cdot \left(A \cdot \frac{\Delta x}{2}\right) = 0$$

i.e.

$$(T_4 - T_5) + \frac{\beta}{2} \cdot (T_4^2 - T_5^2) + \frac{h \cdot (T_a - T_5) \cdot \Delta x}{k_0} + \frac{q_g \cdot (\Delta x)^2}{2 \cdot k_0} = 0$$

i.e.

$$(T_4 - T_5) + \frac{\beta}{2} \cdot (T_4^2 - T_5^2) - 0.3 \cdot T_5 + 29.986 + 5.997 = 0 \quad ...(f)$$

Eq. f is the difference equation for node 5. This equation is also non-linear.

Now, we have got 6 equations, namely Eqs. a, b...f and there are 6 unknown node temperatures. So, solving these 6 coupled equations simultaneously, we get the temperatures T_0, T_1 ... T_6.

We use 'solve block' of Mathcad to solve these equations.

We start with assumed or trial values for all the variables i.e. for the temperatures at nodes 0 to 5. Then, in the solve block, immediately below 'Given' write all the constraint equations. Then, the command 'Find (T_0, T_1, T_2 ... T_5)' immediately gives a vector of temperature values:

$T_0 := 50 \quad T_1 := 50 \quad T_2 := 50 \quad T_3 := 50 \quad T_4 := 50 \quad T_5 := 50$ *(trial values of temperatures)*

Given

$$(T_1 - T_0) + \frac{\beta}{2} \cdot (T_1^2 + T_0^2) + 5.997 = 0 \quad ...(a)$$

$$(T_0 - 2 \cdot T_1 + T_2) + \frac{\beta}{2} \cdot [(T_0)^2 - 2 \cdot (T_1)^2 + T_2)^2] + 11.994 = 0 \quad ...(b)$$

$$(T_1 - 2 \cdot T_2 + T_3) + \frac{\beta}{2} \cdot [(T_1)^2 - 2 \cdot (T_2)^2 + T_3)^2] + 11.994 = 0 \quad ...(c)$$

$$(T_2 - 2 \cdot T_3 + T_4) + \frac{\beta}{2} \cdot [(T_2)^2 - 2 \cdot (T_3)^2 + T_4)^2] + 11.994 = 0 \quad ...(d)$$

$$(T_3 - 2 \cdot T_4 + T_5) + \frac{\beta}{2} \cdot [(T_3)^2 - 2 \cdot (T_4)^2 + T_5)^2] + 11.994 = 0 \quad ...(e)$$

$$(T_4 - T_5) + \frac{\beta}{2} \cdot (T_4^2 - T_5^2) - 0.3 \cdot T_5 + 29.986 + 5.997 = 0 \quad ...(f)$$

Temp: = Find(T_0, T_1, T_2, T_3, T_4, T_5) *('Temp' is the vector containing values of temperatures T_0, T_1, ... T_5)*

Therefore,

$$\text{Temp} = \begin{bmatrix} 414.482 \\ 410.058 \\ 396.709 \\ 374.203 \\ 342.128 \\ 299.853 \end{bmatrix}$$

i.e. Temperatures at different nodes are:

T_0 = 414.482°C T_1 = 410.058°C T_2 = 396.709°C T_3 = 374.203°C T_4 = 342.128°C T_5 = 299.853°C

When there is no analytical solution to compare the results obtained by numerical methods, the number of sub-divisions can be increased and the results obtained with the increased sub-divisions may be compared with the earlier results; and this process may be continued till the difference between the successive results converges to a pre-determined accuracy. Better alternative is to make a heat balance check: In this case, since the left side is insulated, all the heat generated in the slab must be dissipated at the right surface to the fluid by convection. Heat generated per 1 m^2 of area = $(8 \times 10^7) \times (1 \times 0.01) = 8 \times 10^5$ W, and the heat transferred by convection from the right face to the fluid = $h.A.\Delta T$ = 4000 × 1 × (299.853 – 100) = 7.994×10^5 W; i.e. heat generated = heat dissipated by convection.

Example 8.3. A straight fin of rectangular cross-section has length $L = 3$ cm, thickness $t = 0.5$ cm and width $w = 10$ cm. Thermal conductivity of fin material, $k = 20$ W/(mC). Temperature at the base of the fin is $T_0 = 200°C$ and there is negligible heat transfer from the tip of the fin. The fin dissipates heat from its surfaces into the surroundings at 25°C with a heat transfer coefficient of 15 W/(m²C). Using the finite difference method with 10 equally spaced sub-divisions, each of length $\Delta x = 0.3$ cm, determine:

(a) temperatures at the nodes
(b) rate of heat transfer from the fin, and
(c) fin efficiency

Solution.

Data.

$L := 0.03$ m $\quad w := 0.1$ m $\quad t := 0.005$ m $\quad M := 10 \quad k := 20$ W/(mC) $\quad T_0 := 200°C \quad T_a := 25°C$

$h := 15$ W/(m²C) $\quad A_c := w \cdot t$ i.e. $A_c = 5 \times 10^{-4}$ m² $\quad \Delta x := \frac{L}{M} = \frac{0.03}{10}$ i.e. $\Delta x = 0.003$ m

Note that there are 11 nodes, numbered as: 0, 1, 2, 3, 4, ...10. Out of these, nodes '0' and '10' are boundary nodes and the other nodes are internal nodes. Temperature of node '0' is given, i.e. $T_0 = 200°C$.

Fig. Example 8.3 shows the schematic of finite difference nodes for this problem.

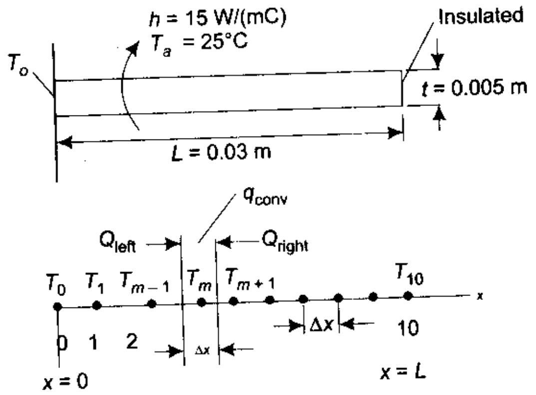


FIGURE Example 8.3 Finite difference nodes for Example 8.3

Difference equations for internal nodes:
Consider a typical internal node 'm' and write an energy balance for the differential volume represented by node 'm'. Remember that all heat flows are into the control volume. There is conduction from nodes $(m - 1)$ and $(m + 1)$ into node m and also there is heat flow by convection from the ambient into the control volume:

$$k \cdot A_c \cdot \left(\frac{T_{m-1} - T_m}{\Delta x}\right) + k \cdot A_c \cdot \left(\frac{T_{m+1} - T_m}{\Delta x}\right) + h \cdot ((2 \cdot w + 2 \cdot t) \cdot \Delta x) \cdot (T_a - T_m) = 0$$

i.e.

$$T_{m-1} - 2 \cdot T_m + T_{m+1} + \frac{2 \cdot h \cdot (w + t) \cdot (\Delta x)^2 \cdot (T_a - T_m)}{k \cdot A_c} = 0 \quad \text{...eqn. (A)}$$

Eq. A gives the finite difference equation for the internal nodes, $m = 1, 2, \dots 9$.

We have:

$$T_{m-1} - 2 \cdot T_m + T_{m+1} + \frac{2 \cdot h \cdot (w + t) \cdot (\Delta x)^2 \cdot (T_a - T_m)}{k \cdot A_c} = 0 \quad \text{...eqn. (A)}$$

Putting $m = 1$ etc.,

Node 1: $T_0 - 2 \cdot T_1 + T_2 - 2.835 \times 10^{-3} \cdot T_1 + 0.071 = 0$ ...(b)

Node 2: $T_1 - 2 \cdot T_2 + T_3 - 2.835 \times 10^{-3} \cdot T_2 + 0.071 = 0$ ...(c)

Node 3: $T_2 - 2 \cdot T_3 + T_4 - 2.835 \times 10^{-3} \cdot T_3 + 0.071 = 0$ ...(d)

Node 4: $T_3 - 2 \cdot T_4 + T_5 - 2.835 \times 10^{-3} \cdot T_4 + 0.071 = 0$ ...(e)

Node 5: $T_4 - 2 \cdot T_5 + T_6 - 2.835 \times 10^{-3} \cdot T_5 + 0.071 = 0$ ...(f)

Node 6: $T_5 - 2 \cdot T_6 + T_7 - 2.835 \times 10^{-3} \cdot T_6 + 0.071 = 0$ ...(g)

Node 7: $T_6 - 2 \cdot T_7 + T_8 - 2.835 \times 10^{-3} \cdot T_7 + 0.071 = 0$ ...(h)

Node 8: $T_7 - 2 \cdot T_8 + T_9 - 2.835 \times 10^{-3} \cdot T_8 + 0.071 = 0$ ...(i)

Node 9: $T_8 - 2 \cdot T_9 + T_{10} - 2.835 \times 10^{-3} \cdot T_9 + 0.071 = 0$ ...(j)

Difference equations for boundary nodes:

For node '0': By data, temperature of node '0' is the temperature of base surface:

i.e. $$T_0 = 200°C \quad \text{...(a)}$$

For node 10: Consider the half-volume surrounding node '10' and write the energy balance, with all heat flow lines into the volume. Remember that heat flow from right of control volume into the node 10 is zero, since the surface is considered as insulated:

$$k \cdot A_c \cdot \left(\frac{T_9 - T_{10}}{\Delta x} \right) + h \cdot \left[(2 \cdot w + 2 \cdot t) \cdot \frac{\Delta x}{2} \right] \cdot (T_a - T_{10}) = 0$$

i.e. $$T_9 - T_{10} + \frac{h \cdot (w + t) \cdot (\Delta x)^2}{k \cdot A_c} \cdot (T_a - T_{10}) = 0$$

i.e. $$T_9 - T_{10} - 1.418 \times 10^{-3} \cdot T_{10} + 0.035 = 0 \quad \text{...(k)}$$

Eqs. a to k give 10 equations for the 10 node temperatures. Solving these equations simultaneously, we get the node temperatures.

We use 'solve block' of Mathcad to solve these equations.

We start with assumed or trial values for all the variables i.e. for the temperatures at nodes 0 to 10. Then, in the solve block, immediately below 'Given' write all the constraint equations. Then, the command 'Find (T_0, T_1, T_2 ... T_{10})' immediately gives a vector of temperature values:

$T_0 := 200$ *(by data)*

$T_1 := 50 \quad T_2 := 50 \quad T_3 := 50 \quad T_4 := 50 \quad T_5 := 50$ *(trial values of temperatures)*

$T_6 := 50 \quad T_7 := 50 \quad T_8 := 50 \quad T_9 := 50 \quad T_{10} := 50$ *(trial values of temperatures)*

Given

$T_0 = 200$ C ...(a)

$T_0 - 2 \cdot T_1 + T_2 - 2.835 \times 10^{-3} \cdot T_1 + 0.071 = 0$ ...(b)

$T_1 - 2 \cdot T_2 + T_3 - 2.835 \times 10^{-3} \cdot T_2 + 0.071 = 0$ ...(c)

$T_2 - 2 \cdot T_3 + T_4 - 2.835 \times 10^{-3} \cdot T_3 + 0.071 = 0$ ...(d)

$T_3 - 2 \cdot T_4 + T_5 - 2.835 \times 10^{-3} \cdot T_4 + 0.071 = 0$ ...(e)

$T_4 - 2 \cdot T_5 + T_6 - 2.835 \times 10^{-3} \cdot T_5 + 0.071 = 0$ ...(f)

$T_5 - 2 \cdot T_6 + T_7 - 2.835 \times 10^{-3} \cdot T_6 + 0.071 = 0$ ...(g)

$T_6 - 2 \cdot T_7 + T_8 - 2.835 \times 10^{-3} \cdot T_7 + 0.071 = 0$ ...(h)

$T_7 - 2 \cdot T_8 + T_9 - 2.835 \times 10^{-3} \cdot T_8 + 0.071 = 0$ ...(i)

$T_8 - 2 \cdot T_9 + T_{10} - 2.835 \times 10^{-3} \cdot T_9 + 0.071 = 0$ ...(j)

$T_9 - T_{10} - 1.418 \times 10^{-3} \cdot T_{10} + 0.035 = 0$ ...(k)

Temp := Find(T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10})...(a)

Therefore, *(Temp. is the vector containing values of temperatures T_0, T_1, ...T_{10})*

Temp =

	0
0	200
1	195.707
2	191.899
3	188.563
4	185.691
5	183.274
6	181.306
7	179.78
8	178.694
9	178.043
10	177.826

i.e. Temperatures at different nodes are:

$T_0 = 200°C$ | $T_6 = 181.306°C$
$T_1 = 195.707°C$ | $T_7 = 179.78°C$
$T_2 = 191.899°C$ | $T_8 = 178.694°C$
$T_3 = 188.563°C$ | $T_9 = 178.043°C$
$T_4 = 185.691°C$ | $T_{10} = 177.826°C$
$T_5 = 183.274°C$

Analytical solution: For the case of a fin, with an insulated end, analytical expression for temperature distribution along the length is:

For fin: $P := 2 \cdot w + 2 \cdot t$ ...perimeter, and $m = \sqrt{\frac{h.p}{k.A_c}}$ i.e. m = 17.748 1/m ...fin parameter

$$T(x) := T_a + (T_0 - T_a) \cdot \frac{\cosh(m \cdot (L - x))}{\cosh(m \cdot L)}$$ *(temperature distribn. in a fin with insulated end)*

	Temperatures from Analytical solution	Temperatures from numerical soluion
Node 0:	$T(0) = 200$	$T_0 = 200°C$
Node 1:	$T(0.003) = 195.706$	$T_1 = 195.707°C$
Node 2:	$T(0.006) = 191.896$	$T_2 = 191.899°C$
Node 3:	$T(0.009) = 188.559$	$T_3 = 188.563°C$
Node 4:	$T(0.012) = 185.686$	$T_4 = 185.691°C$
Node 5:	$T(0.015) = 183.269$	$T_5 = 183.274°C$
Node 6:	$T(0.018) = 181.3$	$T_6 = 181.306°C$
Node 7:	$T(0.021) = 179.775$	$T_7 = 179.78°C$
Node 8:	$T(0.024) = 178.688$	$T_8 = 178.694°C$
Node 9:	$T(0.027) = 178.038$	$T_9 = 178.043°C$
Node 10:	$T(0.03) = 177.821$	$T_{10} = 177.826°C$

We observe that values of temps. obtained by numerical methods match extremely well with the values obtained by 'exact' analytical solution, i.e. with only 10 equal divisions of the fin length, we get very accurate solution by numerical method.

Heat transferred by the fin, Q_{fin}:

Q_{fin} must be equal to the amount of heat entering into the fin at its base.

Write the heat balance for the half-volume around node '0':

$$Q_{fin} + \left[\frac{\frac{T_1 - T_0}{\Delta x}}{k \cdot A_c} \right] + h \cdot (2 \cdot w + 2 \cdot t) \cdot \frac{\Delta x}{2} \cdot (T_a - T_0) = 0$$

i.e.
$$Q_{fin} := \left[\frac{\frac{T_0 - T_1}{\Delta x}}{k \cdot A_c} \right] - h \cdot (2 \cdot w + 2 \cdot t) \cdot \frac{\Delta x}{2} (T_a - T_0)$$

i.e. $Q_{fin} = 15.137$ **W.**

Fin efficiency, η_f:

Fin efficiency is the ratio of actual heat transferred by the fin to the maximum heat that would be transferred if the entire fin surface were at the base temp.

$$\eta_f = \frac{Q_{fin}}{Q_{max}}$$

$Q_{max} := h \cdot (2 \cdot w + 2 \cdot t) \cdot L \cdot (T_0 - T_a)$ W *(maximum heat transfer, if the entire fin were at base temperature)*

i.e. $Q_{max} = 16.538$ W *(maximum heat transfer by fin)*

Therefore, $\eta_f := \frac{Q_{fin}}{Q_{max}}$ ***(fin efficiency)***

i.e. $\eta_f = 0.915 =$ **91.5%**

8.4 Methods of Solving a System of Simultaneous, Algebraic Equations

From what we have studied so far, it is clear that while solving steady state heat conduction problems by finite difference formulation, we get a set of algebraic equations by applying the energy balance to the various nodes and this set has to be solved simultaneously to obtain the temperatures at different nodes. There are many equation solvers and powerful software using which one can obtain the solution easily without knowing the intricacies of the methods involved. While solving problems, you might have noticed that we used Mathcad, in which solving even non-linear algebraic equations was extremely simple. Also, ready computer programs and subroutines are available for users, who have to simply change one or two lines of the program to adapt the solution for their particular problem. For solving simultaneous, linear algebraic equations, subroutines such as LEQT1F, LEQT1B, LEQT2F, LEQT2B supplied by the International Mathematical and Statistical Libraries (IMSL) have been popular in scientific community.

Still, it is worthwhile to know the basics of different methods involved in solving a set of algebraic equations. We shall briefly present a few methods:

(i) Relaxation method

(ii) Direct methods: (a) Gaussian elimination , and (b) Matrix inversion

(iii) Iterative methods, e.g. Gauss – Siedel iteration method

(i) Relaxation method This is basically a trial and error solution and does not require a computer. But, it is practicable to use only when the number of equations is small, say, less than 10. As an example, consider a set of following three algebraic equations:

$$a_1 \cdot x + b_1 \cdot y + c_1 \cdot z = 0$$
$$a_2 \cdot x + b_2 \cdot y + c_2 \cdot z = 0$$
$$a_3 \cdot x + b_3 \cdot y + c_3 \cdot z = 0$$

The coefficients a_1, b_1, ..., c_1 etc. are known, and our aim is to solve this set for x, y and z. Then, the 'Relaxation technique' consists of the following steps:

(a) To start with, assume values for x, y and z.

(b) Since the assumed values are certainly likely to be in error, each of the above equations will not be zero, but equal to some residual values R_1, R_2 and R_3:

$$a_1 \cdot x + b_1 \cdot y + c_1 \cdot z = R_1$$
$$a_2 \cdot x + b_2 \cdot y + c_2 \cdot z = R_2$$
$$a_3 \cdot x + b_3 \cdot y + c_3 \cdot z = R_3$$

(c) Our aim is to reduce R_1, R_2 and R_3 to zero by suitably varying the assumed values of x, y and z, by trial and error. This is done systematically, by first setting up a 'unit change table', i.e. a table showing the change in the values of residuals for unit change in x, y and z.

(d) Set up a 'Relaxation table' wherein you begin with the initially assumed values of x, y and z and the resultant residuals. Then, start 'relaxing' the largest residual by suitably changing the value of x, y or z, taking guidance from the 'unit table' already set up.

(e) Continue the procedure till all the residuals are relaxed to zero.

Obviously, this procedure is slow and time consuming and cannot be used when the number of equations to be solved is large.

(ii) Direct methods Direct methods have a fixed number of well defined steps to systematically solve the equations for the unknown values. However, they consume more of computer memory and time compared to iterative methods, and are suitable for comparatively small number of equations. Under 'direct methods', we shall study two methods: (a) Gaussian elimination method, and (b) Matrix inversion method:

(a) Gaussian elimination method In this method, used for solution of a system of linear algebraic equations, one of the unknowns is eliminated systematically in each step, and at the end of the elimination process, the last equation involves only one unknown, and then the remaining unknowns are obtained one by one by 'back substitution'. To make the process clear, let us consider a simple system of three algebraic equations, as given below:

$$x + 2 \cdot y + 3 \cdot z = 33 \quad ...(a)$$
$$x - 4 \cdot y + z = -11 \quad ...(b)$$
$$3 \cdot x + y + z = 18 \quad ...(c)$$

Now, we 'triangularize' the given set of equations by repeated application of three basic row operations:

(i) multiplication of a row by a constant (ii) adding one row to another row, and (iii) interchange of two rows.

In the above, use Eq. a to elimlnate x from Eqs. b and c, by adding –1 times (a) to (b) and by adding –3 times (a) to (c). We get:

$$x + 2 \cdot y + 3 \cdot z = 33 \quad \text{...(a')}$$
$$-6 \cdot y - 2 \cdot z = 44 \quad \text{...(b')}$$
$$-5 \cdot y - 8 \cdot z = 81 \quad \text{...(c')}$$

Next, eliminate y from Eq. c′ by multiplying Eq. b′ by –5/6 and adding to Eq. c′:

We get:

$$x + 2 \cdot y + 3 \cdot z = 33$$
$$6 \cdot y + 2 \cdot z = 44$$
$$z = 7$$

Above set of equations is known as 'triangularized set' of equations.

Having obtained the value of z, now back-substitute in the previous equations to get value of y as $y = 5$, and one more 'back-substitution' in the preceding equation gives the value of x as $x = 2$.

Since we had only three equations in the above set, we could do the elimination or triangularization by hand. However, Gaussian elimination method for a system of large number of equations is done with a computer, using matrix notation to represent the equation. Coefficients constitute a square matrix called 'coefficient matrix' and the constant terms are stored in a vector called 'right hand side vector'. Computation sub-routines normally combine these two into a single 'augmented matrix' and the above procedure is done by the computer program to eliminate the terms below the main diagonal of the augmented matrix. This results in a matrix of 'upper diagonal form'. Then, back-substitution is performed by the program systematically to get the solution.

So, for example, in the above set of equations, the augmented matrix will be:

$$\begin{bmatrix} 1 & 2 & 3 & 33 \\ 1 & -4 & 1 & -11 \\ 3 & 1 & 1 & 18 \end{bmatrix}$$

Now, resorting to already mentioned row operations on this matrix, elements under the main diagonal are eliminated and the upper diagonal form of the matrix is obtained as:

$$\begin{bmatrix} 1 & 2 & 3 & 33 \\ 0 & 6 & 2 & 44 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Last row means that $z = 7$. Now, back-substitution is done to get values of y and x. Gaussian elimination method is conveniently programmed in a computer and ready subroutines are available to solve a set of N linear algebraic equations simultaneously.

(b) Matrix inversion method In this method, the set of equations is written in the following matrix form:

$$[A]\,[T] = [B], \text{ where}$$

[A] is the coefficient matrix, [T] is the vector of temperatures to be found out, and [B] is the vector of constants (RHS) of the equations. Solution of this system by matrix inversion method is given by:

$$[T] = [A]^{-1}\,[B], \text{ where } [A]^{-1} \text{ is the inverse of matrix } [A].$$

Matrix inversion is performed generally by using readily available computer subroutines. In Mathcad, inverse of a matrix A is obtained in a single step by the command $A^{-1} =$.

For the problem illustrated above, we have:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -4 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 33 \\ -11 \\ 18 \end{bmatrix}$$

and,

$$A^{-1} \rightarrow \begin{bmatrix} \frac{-5}{38} & \frac{1}{38} & \frac{7}{19} \\ \frac{1}{19} & \frac{-4}{19} & \frac{1}{19} \\ \frac{13}{38} & \frac{5}{38} & \frac{-3}{19} \end{bmatrix}$$ *(inverse of A, from Mathcad)*

Therefore,

$$T := A^{-1} \cdot B$$ *(T is the vector containing x, y, z as its elements)*

i.e. $$T = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

which means that $x = 2$, $y = 5$ and $z = 7$.

This result is the same as obtained earlier.

Once again, when the number of equations is relatively large, this is not a preferred method, from the point of view of computer memory and storage.

(iii) Gauss–Siedel iteration method Iteration methods are used when the number of algebraic equations to be solved is relatively large. Gauss–Siedel iteration (also called Liebmann iteration) method is one of the most popular iteration methods because of its simplicity. The method involves the following steps:

(a) Solve each equation for one of the unknowns, i.e. write each unknown in terms of other unknowns

(b) Assume guess values for all unknowns, and from the equations developed in step (a), compute the unknowns, each time using the most recently computed values for the unknowns in each equation

(c) Repeat this procedure until the successive values of an unknown converge to a specified accuracy.

To illustrate this procedure, let us consider the example given below. We have a set of equations as follows:

$$3 \cdot x - y + 3 \cdot z = 0 \quad \text{...(a)}$$

$$-x + 2 \cdot y + z = 3 \quad \text{...(b)}$$

$$2 \cdot x - y - z = 2 \quad \text{...(c)}$$

Now, write each equation for one of the unknowns. i.e.

$$x = \frac{y - 3 \cdot z}{3}$$

$$y = \frac{(3 + x - z)}{2}$$

$$z = -2 + 2 \cdot x - y$$

Now, assume guess values for x, y and z. Say, $x = 1$, $y = 1$ and, $z = 1$. These are the 'zeroth' iteration values.

With these guess values, begin the iteration and in each equation, use the latest values of unknowns as available. So, after 'first' iteration we have:

$x = 1$ $y = 1$ $z = 1$ *(inital guess values)*

$x = \frac{y - 3 \cdot z}{3}$ i.e. $x = -0.667$ *(with $y = 1$, $z = 1$)*

$y = \frac{(3 + x - z)}{2}$ i.e. $y = 0.667$ *(with $x = -0.667$, $z = 1$)*

$z = -2 + 2 \cdot x - y$ i.e. $z = -4$ ...with $x = -0.667$, $y = 0.667$

Now, for the 'second' iteration, continue the procedure, with the latest values of unknowns. We get:

$x = -0.667$ $y = 0.667$ $z = -4$ *(next guess values from previous iteration)*

$x = \frac{y + 3 \cdot z}{3}$ i.e. $x = 4.222$ *(with $y = 0.667$, $z = -4$)*

$y = \frac{(3 + x - z)}{2}$ i.e. $y = 5.611$ (*with* $x = 4.222$, $z = -4$)

$z = -2 + 2 \cdot x - y$ i.e. $z = 0.833$ (*with* $x = 4.222$, $y = 5.611$)

For the 'third' iteration, take $x = 4.222$, $y = 5.611$ and $z = 0.833$, and continue. This process is programmed easily in a computer and the results normally converge within about 100 iterations. Of course, we can also instruct the program to stop when the difference between successive values of unknowns converge to a pre-determined accuracy.

A simple Mathcad program to perform the above iteration is shown below. It does the iteration 100 times. Final values of x, y and z are returned as a vector R.

$$
\begin{aligned}
R := \;& x_0 \leftarrow 1 \\
& y_0 \leftarrow 1 \\
& z_0 \leftarrow 1 \\
& \text{for } i \in 0..100 \\
& \quad x_{i+1} \leftarrow \frac{y_i - 3 \cdot z_i}{3} \\
& \quad y_{i+1} \leftarrow \frac{(3 + x_{i+1} - z_i)}{2} \\
& \quad z_{i+1} \leftarrow (-2 + 2 \cdot x_{i+1} - y_{i+1}) \\
& \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{bmatrix}
\end{aligned}
$$

And, $$R = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

which means that $x = 2$, $y = 3$ and $z = -1$.

In the above program, LHS defines a vector R. On the RHS, there are 10 lines. First three lines assign the initial guess values for x, y and z. Next 4 lines show the 'for loop', for 100 iterations, where in x, y and z are calculated, each time using the latest available values of unknowns. Next 3 lines constitute the latest values of x, y and z which are stored as the elements of the vector R.

It is interesting to note that in the above program, if iteration is carried out only for 5, 10, 20, 50 and 100 loops (by changing the 4th line), following are the results:

After 5 iterations	After 10 iterations	After 20 iterations	After 50 iterations	After 100 iterations
$R = \begin{bmatrix} 1.755 \\ 2.718 \\ -1.208 \end{bmatrix}$	$R = \begin{bmatrix} 1.987 \\ 2.982 \\ -1.008 \end{bmatrix}$	$R = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$	$R = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$	$R = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

i.e. even with only 10 iterations we are very close to the final result. By the time 20 iterations are over, solution has already converged to the final result.

It is stated that for steady state heat conduction problems, Gauss–Siedel iteration process is inherently stable and always converges into a solution.

Note: Of course, above program can be further refined to stop when the successive values of x, y and z differ by a pre-determined small value ε. (say, $\varepsilon = 0.001$).

Above program in Mathcad is shown only to illustrate the procedure of iterative solution. While actually using Mathcad, we would use the 'solve block' (which also follows an iterative algorithm), as follows:

$x := 0 \qquad y := 0 \qquad z := 0$ *(guess values)*

Given

$$3\cdot x - y + 3\cdot z = 0 \quad ...(a)$$
$$-x + 2\cdot y + z = 3 \quad ...(b)$$
$$2\cdot x - y - z = 2 \quad ...(c)$$

$$\text{Find }(x, y, z) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

You may put any guess value to start with; it makes no difference on the final result. However, it is essential that each unknown is assigned some guess value to start with.

Accuracy of the solutions Some comments on the accuracy of finite difference solutions are appropriate. We noted earlier that in solving heat conduction problems by finite difference methods, accuracy improves as the number of nodes is made larger. However, this would mean that a larger number of algebraic equations have to be solved simultaneously. This situation has following inherent drawbacks: the computer memory required increases and also, more importantly, the round off errors in successive calculations increase since they are cumulative. Therefore, one should start with a coarse mesh and then gradually refine it depending upon the accuracy of final results required. Note that for the normal problems encountered in practice, a coarse mesh generally gives results of acceptable accuracy; remember that, anyway, there are uncertainties in the values of thermal properties and heat transfer coefficients available to the designer.

8.5 One-dimensional, Steady State Conduction in Cylindrical Systems

We shall now develop finite difference formulation by energy balance method, for one-dimensional, steady state heat conduction in cylindrical coordinates.

Consider a long, solid cylinder of radius R in which the heat flow is only in the radial direction. Let the rate of internal heat generation be $q_g(\text{W/m}^3)$. The region from $r = 0$ to $r = R$ is divided into M sub-regions, each of thickness $\Delta r = R/M$. Therefore, there are

$(M + 1)$ nodes, numbered as 0, 1, 2, ..., M. See Fig. 8.5.

Writing an energy balance for the volume element around node 'm', remembering that all heat flows are into the volume, we get; in steady state:

Rate of energy flowing into the volume from left + Rate of energy flowing into the volume from right + Rate of heat generated in the volume = 0.

Substituting the values,

$$\frac{\dfrac{T_{m-1} - T_m}{\Delta r}}{2\cdot\pi\cdot\left(m\cdot\Delta r - \dfrac{\Delta r}{2}\right)\cdot L\cdot k} + \frac{\dfrac{T_{m+1} - T_m}{\Delta r}}{2\cdot\pi\cdot\left(m\cdot\Delta r + \dfrac{\Delta r}{2}\right)\cdot L\cdot k} + (2\cdot\pi\cdot m\cdot\Delta r\cdot\Delta r)\cdot L\cdot q_m = 0$$

First term in the above equation is the heat flowing into node 'm' from node '$(m - 1)$'. Denominator of the first term is the thermal resistance between 'm' and '$(m - 1)$'; it is written in the form $(L/k\,A)$ where A is the mean area i.e. area of the plane mid-way between nodes 'm' and '$(m - 1)$'. This form of thermal resistance (as if for a plane wall), is alright for the cylindrical system when $\Delta r << R$, which is generally the case. Also, this makes the equation simpler. Similarly, the second term in the above equation is the heat flowing into node 'm' from node '$(m + 1)$'. The third term gives the heat generated in the elemental volume. L is the length of the cylinder and q_m is the heat generation rate per unit volume for the elemental volume (= q_g, generally).

Simplifying the above equation, we get:

$$\left(1 - \frac{1}{2\cdot m}\right)\cdot T_{m-1} - 2\cdot T_m + \left(1 + \frac{1}{2\cdot m}\right)\cdot T_{m+1} + \frac{(\Delta r)^2\cdot q_m}{k} = 0 \quad ...(8.32)$$

Eq. 8.32 is the finite difference equation for internal nodes i.e. for nodes 1, 2, ...,$(M - 1)$, with constant thermal conductivity and internal heat generation.

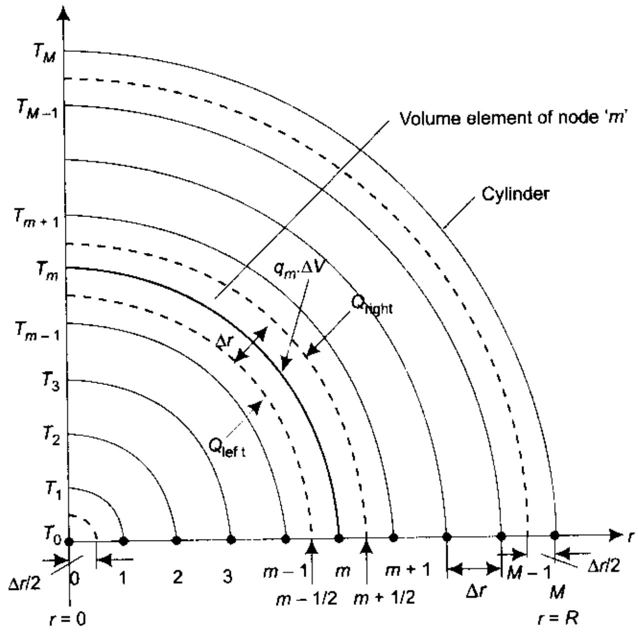


FIGURE 8.5 Finite difference formulation in a cylindrical/spherical system

At the centre: i.e. at $r = 0$:
Writing the energy balance for the half-volume (of thickness $\Delta r/2$) around node '0', we get:

$$\frac{\dfrac{T_1 - T_0}{\Delta r}}{2\cdot\pi\cdot\dfrac{\Delta r}{2}\cdot L\cdot k} + \pi\cdot\left(\frac{\Delta r}{2}\right)^2\cdot L\cdot q_0 = 0$$

In the above, first term is the heat conduction rate from node '1' to node '0' and the second term is the heat generation term. q_0 is the heat generation rate per unit volume at node '0' (= q_g, generally). Simplifying the above equation, we get:

$$4\cdot(T_1 - T_0) + \frac{(\Delta r)^2\cdot q_0}{k} = 0 \qquad ...(8.33)$$

Eq. 8.33 gives the finite difference equation for the centre node '0', with constant thermal conductivity and internal heat generation.
At the periphery: i.e. at node 'M':
As in the earlier cases, here too, finite difference equation is obtained by applying the energy balance to the half-volume around node 'M'. Of course, the nature of equation depends on the boundary condition, i.e. if it is prescribed temperature, or prescribed heat flux or convection boundary condition. For convection boundary conditions, where heat transfer from the periphery is with an ambient at temperature T_a with a heat transfer coefficient of h, energy balance around node 'M', gives:

$$\frac{\dfrac{T_{M-1} - T_M}{\Delta r}}{2\cdot\pi\cdot\left(M\cdot\Delta r - \dfrac{\Delta r}{2}\right)\cdot L\cdot k} + (2\cdot\pi\cdot M\cdot\Delta r\cdot L)\cdot h\cdot(T_a - T_M) + 2\cdot\pi\cdot M\cdot\Delta r\cdot\frac{\Delta r}{2}\cdot L\cdot q_M = 0$$

In the above, first term is the heat conduction rate from node '$(M - 1)$' to node 'M' and the second term is the convective heat transfer between the periphery and the ambient, and the third term is the heat generation term.

q_M is the heat generation rate per unit volume at node 'M' (= q_g, generally). Simplifying the above equation, we get:

$$\left(1-\frac{1}{2\cdot M}\right)\cdot T_{M-1}-\left[\left(1-\frac{1}{2\cdot M}\right)+\frac{\Delta r\cdot h}{k}\right]\cdot T_M+\frac{\Delta r\cdot h}{k}\cdot T_a+\frac{(\Delta r)^2\cdot q_M}{2\cdot k}=0 \quad ...(8.34)$$

Eq. 8.34 gives the finite difference equation for the boundary node 'M', with convection conditions, constant thermal conductivity and internal heat generation.

Example 8.4. A nuclear fuel element is in the form of a hollow cylinder insulated at the inner surface. Its inner and outer radii are 5 cm and 10 cm respectively. The outer surface gives heat to a fluid at 50°C where the unit surface conductance is 100 W/(m^{2}K). k of the material is 50 W/(mK). Dividing the shell into 5 equal sub-regions, find the temperatures at different nodes. What is the maximum temperature in the system? Given: Rate of heat generation in the fuel element is 3.796 × 10^5 W/m^3.

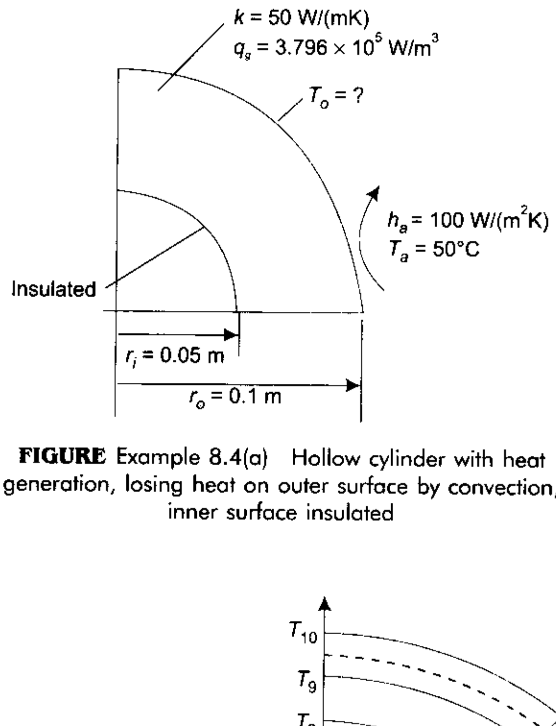


FIGURE Example 8.4(a) Hollow cylinder with heat generation, losing heat on outer surface by convection, inner surface insulated

Solution. See Fig. Example 8.4 (a).

Data:

$r_i := 0.05$ m $\quad r_o := 0.1$ m $\quad k := 50$ W/mK

$T_a := 50$°C $\quad h := 100$ W/(m^{2}K) $\quad L := 1$ m

$q_g := 3.796\times10^5$ W/m^3 $\quad M := 10$

$\Delta r := \frac{r_o - r_i}{5}$ i.e. $\Delta r = 0.01$ m

Fig. 8.4 (b) shows the schematic for finite difference representation:

Note: Temperature on inner surface is the maximum temperature since inner surface is insulated.

Here, nodes 5 and 10 are boundary nodes and nodes 6, 7, 8 and 9 are internal nodes. Node 5 is on the insulated surface and node 10 is on the outer surface with convection boundary condition. By this system of numbering, r coordinate of node 'm' is $m.\Delta r$ and M = 10 = no. of subdivisions in the outer radius.

Finite difference equations for internal nodes:

i.e. for nodes 6 to 9, we can apply Eq. 8.32, viz.

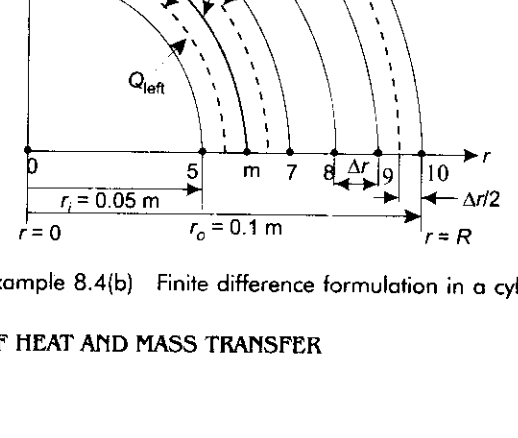


FIGURE Example 8.4(b) Finite difference formulation in a cylindrical system

$$\left(1-\frac{1}{2\cdot m}\right)\cdot T_{m-1}-2\cdot T_m+\left(1+\frac{1}{2\cdot m}\right)\cdot T_{m+1}+\frac{(\Delta r)^2\cdot q_m}{k}=0 \quad ...(8.32)$$

Putting $m = 6.....9$, in the above equation we get difference equations for those respective nodes:

$m = 6$: $0.917\cdot T_5 - 2\cdot T_6 + 1.083\cdot T_7 + 0.759 = 0$ ...(b)

$m = 7$: $0.929\cdot T_6 - 2\cdot T_7 + 1.071\cdot T_8 + 0.759 = 0$ ...(c)

$m = 8$: $0.938\cdot T_7 - 2\cdot T_8 + 1.063\cdot T_9 + 0.759 = 0$ ...(d)

$m = 9$: $0.944\cdot T_8 - 2\cdot T_9 + 1.056\cdot T_{10} + 0.759 = 0$ ...(e)

Finite difference equations for boundary nodes:

For node 5 Temperature on the insulated surface is maximum. Difference equation for node 5 is obtained by writing heat balance on the half-volume around node 5, with all the heat flow lines going into the volume. From the left there is no heat flow since the surface is insulated. So, there is heat flow only from right and there is heat generation term:

i.e.
$$\frac{\dfrac{T_6-T_5}{\Delta r}}{2\cdot\pi\cdot\left(5\cdot\Delta r+\dfrac{\Delta r}{2}\right)\cdot L\cdot k}+\pi\cdot\left[\left(5\cdot\Delta r+\frac{\Delta r}{2}\right)^2-(5\cdot\Delta r)^2\right]\cdot L\cdot q_g=0$$

In the above equation first term gives the heat flow into the half-volume and the second term is the heat generation term for the half-volume.

Simplifying, we get:

$$(T_6-T_5)\cdot\left[2\cdot\pi\cdot\left(5\cdot\Delta r+\frac{\Delta r}{2}\right)\cdot k\right]+\Delta r\cdot\pi\cdot\left[\left(5\cdot\Delta r+\frac{\Delta r}{2}\right)^2-(5\cdot\Delta r)^2\right]\cdot q_g=0$$

i.e.
$$(T_6-T_5)\cdot 17.279+6.261=0 \quad ...(a)$$

For node 10 We can use Eq. 8.34; but, in this case, we can straightaway get the temperature at node 10 by a heat balance at the outer surface, as follows:

All heat generated in the shell goes to the outer surface since the inner surface is insulated. Writing the heat balance at the outer surface,

Heat generation in the shell = heat transferred by convection at the outer surface

i.e.
$$\pi\cdot(r_o^2-r_i^2)\cdot L\cdot q_g=h\cdot(2\cdot\pi\cdot r_o\cdot L)\cdot(T_{10}-T_a)$$

i.e.
$$T_{10}=\frac{\pi\cdot(r_o^2-r_i^2)\cdot L\cdot q_g}{h\cdot(2\cdot\pi\cdot r_o\cdot L)}+T_a$$

i.e.
$$T_{10}=192.35°\text{C} \text{ ...eqn. (f)}$$ ...*temperature at node 10*

Now, we have 6 equations a to f. Solving them simultaneously, we get temperatures at 6 nodes, i.e. nodes 5 to 10.

We use 'solve block' of Mathcad to solve this set of equations. Start with guess values for all unknown temperatures and immediately below 'Given', type the constraint equations. Then, the command 'Find (T_5, ..., T_{10})' gives the temperatures immediately:

$T_5 := 50 \quad T_6 := 50 \quad T_7 := 50 \quad T_8 := 50 \quad T_9 := 50 \quad T_{10} := 192.35$ (*guess values of temperatures*)

Given

$(T_6 - T_5)\cdot 17.279 + 6.261 = 0$ ...(a)

$0.917\cdot T_5 - 2\cdot T_6 + 1.083\cdot T_7 + 0.759 = 0$ ..(b)

$0.929\cdot T_6 - 2\cdot T_7 + 1.071\cdot T_8 + 0.759 = 0$ ...(c)

$0.938\cdot T_7 - 2\cdot T_8 + 1.063\cdot T_9 + 0.759 = 0$ (d)

$0.944\cdot T_8 - 2\cdot T_9 + 1.056\cdot T_{10} + 0.759 = 0$ ...(e)

$T_{10} = 192.35$ ...(f)

Temp. := Find(T_5, T_6, T_7, T_8, T_9, T_{10}) (*define vector Temperature in which node temps. are stored*)

Therefore,
$$\text{Temp}=\begin{bmatrix}200.371\\200.008\\199.001\\197.418\\196.122\\192.35\end{bmatrix}$$

i.e. The temperatures at different nodes are:

$T_5 = 200.371°C \quad T_6 = 200.008°C \quad T_7 = 199.001°C \quad T_8 = 197.418°C \quad T_9 = 195.122°C \quad T_{10} = 192.35°C$

Check: We can check these results with the analytical solution. This is a problem of a cylindrical shell with the inside surface insulated and outside surface losing heat by convection; in that case, temperature distribution is given by:

$$T(r): = \left[T_a + \frac{q_g \cdot (r_o^2 - r_i^2)}{2 \cdot h \cdot r_o} + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r}\right) - \left(\frac{r}{r_i}\right)^2\right]\right] \quad ...(5.29)$$

(Analytical solution for temperature distribution)

Substitute different values for r and get temperatures at corresponding nodes:

	Analytical solution	Numerical soluion with 5 nodes
Node 5:	$T(0.05) = 200.007$	$T_5 = 200.371°C$
Node 6:	$T(0.06) = 199.649$	$T_6 = 200.008°C$
Node 7:	$T(0.07) = 198.645$	$T_7 = 199.001°C$
Node 8:	$T(0.08) = 197.065$	$T_8 = 195.418°C$
Node 9:	$T(0.09) = 194.956$	$T_9 = 195.122°C$
Node 10:	$T(0.1) = 192.35$	$T_{10} = 192.35°C$

Just for comparison, **if the shell is divided into 10 nodes,** (numbered from 10 to 20), each of radial thickness 0.005 m, following will be the result, in comparison to the analytical results:

$$T(r) = \left[T_a + \frac{q_g \cdot (r_o^2 - r_i^2)}{2 \cdot h \cdot r_o} + \frac{q_g \cdot r_i^2}{4 \cdot k} \cdot \left[\left(\frac{r_o}{r_i}\right)^2 - 2 \cdot \ln\left(\frac{r_o}{r}\right) - \left(\frac{r}{r_i}\right)^2\right]\right]$$

(Analytical solution for temperature distribution)

Substitute for r and get temperatures at different nodes:

	Analytical solution	Numerical soluion with 10 nodes
Node 10:	$T(0.05) = 200.007$	200.024°C
Node 11:	$T(0.055) = 199.915$	199.931°C
Node 12:	$T(0.06) = 199.649$	199.665°C
Node 13:	$T(0.065) = 199.223$	199.237°C
Node 14:	$T(0.07) = 198.645$	198.658°C
Node 15:	$T(0.075) = 197.924$	197.936°C
Node 16:	$T(0.08) = 197.065$	197.076°C
Node 17:	$T(0.085) = 196.075$	196.083°C
Node 18:	$T(0.09) = 194.956$	194.962°C
Node 19:	$T(0.095) = 193.714$	193.717°C
Node 20:	$T(0.01) = 192.35$	192.35°C

It may be noted that the values by finite difference methods match extremely well with the analytical results.

8.6 One-dimensional, Steady State Conduction in Spherical Systems

We shall now develop finite difference formulation by energy balance method, for one-dimensional, steady state heat conduction in spherical coordinates.

Consider a solid sphere of radius R in which the heat flow is only in the radial direction. Let the rate of internal heat generation be q_g (W/m^3). The region from $r = 0$ to $r = R$ is divided into M sub-regions, each of thickness $\Delta r = R/M$. Therefore, there are $(M + 1)$ nodes, numbered as 0, 1, 2, ..., M. See Fig. 8.5.

Writing an energy balance for the volume element around node 'm', remembering that all heat flows are into the volume, we get; in steady state:

Rate of energy flowing into the volume from left + Rate of energy flowing into the volume from right + Rate of heat generated in the volume = 0.

Substituting the values,

$$\frac{\dfrac{T_{m-1} - T_m}{\Delta r}}{4 \cdot \pi \cdot \left(m \cdot \Delta r - \dfrac{\Delta r}{2}\right)^2 \cdot k} + \frac{\dfrac{T_{m+1} - T_m}{\Delta r}}{4 \cdot \pi \cdot \left(m \cdot \Delta r + \dfrac{\Delta r}{2}\right)^2 \cdot k} + [4 \cdot \pi \cdot (m \cdot \Delta \cdot r)^2 \Delta r] \cdot q_m = 0 \quad ...(8.35)$$

First term in the above equation is the heat flowing into node 'm' from node '$(m-1)$'. Denominator of the first term is the thermal resistance between 'm' and '$(m-1)$'; it is written in the form (L/kA) where A is the mean area i.e. area of the plane mid-way between nodes 'm' and '$(m-1)$'. This form of thermal resistance (as if for a plane wall), is alright for the spherical system when $\Delta r << R$, which is generally the case. Also, it makes the equation simpler. Similarly, the second term in the above equation is the heat flowing into node 'm' from node '$(m+1)$'. The third term gives the heat generated in the elemental volume. q_m is the heat generation rate per unit volume for the elemental volume ($= q_g$, generally).

Simplifying the above equation, we get:

$$\left(1-\frac{1}{2\cdot m}\right)^2\cdot(T_{m-1}-T_m)+\left(1+\frac{1}{2\cdot m}\right)^2\cdot(T_{m+1}-T_m)+\frac{(\Delta r)^2\cdot q_m}{k}=0 \qquad ...(8.36)$$

Eq. (8.36) gives finite difference equations for internal nodes, i.e. nodes 1, 2, ..., $(M-1)$.

At the centre, $r = 0$:

Applying the energy balance to the half-volume around node '0',

$$\frac{\dfrac{T_1-T_0}{\Delta r}}{4\cdot\pi\cdot\left(\dfrac{\Delta r}{2}\right)^2\cdot k}+\frac{4}{3}\cdot\pi\cdot\left(\frac{\Delta r}{2}\right)^3\cdot q_0=0$$

Simplifying,

$$6\cdot(T_1-T_0)+\frac{(\Delta r)^2\cdot q_0}{k}=0 \qquad ...(8.37)$$

Eq. 8.37 gives finite difference equation for the centre i.e. node '0'.

For the boundary node 'M':

Difference equation for boundary node 'M' is written in the same manner as was done for other nodes, i.e. by writing an energy balance on the half-volume around node 'M'. The nature of relation obtained will depend upon the boundary condition i.e. prescribed temperature, prescribed heat flux, or convection boundary condition.

As an example, let us write the difference equation for node 'M' when the convection conditions prevail at the boundary. Let there be heat transfer at the boundary with a fluid flowing at a temperature of T_a with a heat transfer coefficient of 'h'. Then, writing an energy balance for the half-volume around node 'M', we get:

$$\frac{\dfrac{T_{M-1}-T_M}{\Delta r}}{4\cdot\pi\cdot\left(M\cdot\Delta r-\dfrac{\Delta r}{2}\right)^2\cdot k}+4\cdot\pi\,(M\cdot\Delta\cdot r)^2\cdot h(T_a-T_M)+4\cdot\pi\cdot(M\cdot\Delta r)^2\cdot\frac{\Delta r}{2}\cdot q_M=0 \qquad ...(8.38)$$

In the above, first term is the heat conduction rate from node '$(M-1)$' to node 'M' and the second term is the convective heat transfer between the outer surface and the ambient, and the third term is the heat generation term. q_M is the heat generation rate per unit volume at node 'M' ($= q_g$, generally). Simplifying the above equation, we get:

$$\left(1-\frac{1}{2\cdot M}\right)^2\cdot T_{M-1}-\left[\left(1-\frac{1}{2\cdot M}\right)^2+\frac{\Delta r\cdot h}{k}\right]\cdot T_M+\frac{\Delta r\cdot h}{k}\cdot T_a+\frac{(\Delta r)^2\cdot q_M}{2\cdot k}=0 \qquad ...(8.39)$$

Eq. 8.39 gives the difference equation for the boundary node 'M' when convection conditions prevail at the boundary.

Example 8.5. A solid sphere of radius, $R = 10$ mm and $k = 18$ W/(mC) has an uniform heat generation rate of 2×10^6 W/m^3. Heat is conducted away at its outer surface to ambient air at 20°C by convection, with a heat transfer coefficient of 2000 W/(m^{2}C). Using numerical method and dividing the radius into 10 equal sub-divisions,

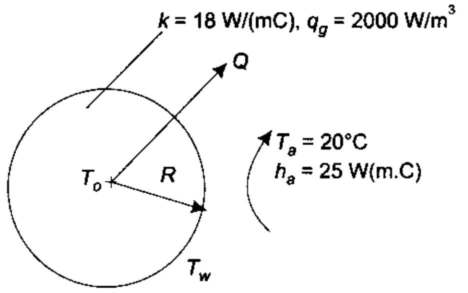


FIGURE Example 8.5 Solid sphere with heat generation

(i) determine the steady state temperature at the centre and outer surface of the sphere.

(ii) draw the temperature profile along the radius.

Solution. Let us now solve this problem numerically, dividing the radius into 10 equal sub-divisions, i.e. $M = 10$. So, there are 11 nodes, marked as 0, 1, ..., 10.

Data:

$R := 0.01$ m $\quad h_a := 2000$ W/(m²K) $\quad k := 18$ W/(mK)

$T_a := 20$°C $\quad q_g := 2 \times 10^6$ W/m³ $\quad M := 10 \quad \Delta r := 0.001$ m

Refer to Fig. 8.5 for finite difference scheme of nodes for this problem.

Node '0' is the centre node, and node '10' is the boundary node. Nodes 1, 2, ..., 9 are the internal nodes. There is convection condition at the boundary.

For internal nodes 1, 2...,9: We apply Eq. 8.36 for internal nodes:

$$\left(1-\frac{1}{2\cdot m}\right)^2\cdot(T_{m-1}-T_m)+\left(1+\frac{1}{2\cdot m}\right)^2\cdot(T_{m+1}-T_m)+\frac{(\Delta r)^2\cdot q_m}{k}=0 \quad ...(8.36)$$

Putting m = 1, 2,...9, we get difference equations for those respective nodes:

m = 1, Node 1: $\quad 0.25\cdot(T_0-T_1)+2.25\cdot(T_2-T_1)+0.111=0$ ...(b)

m = 2, Node 2: $\quad 0.563\cdot(T_1-T_2)+1.563\cdot(T_3-T_2)+0.111=0$ ...(c)

m = 3, Node 3: $\quad 0.694\cdot(T_2-T_3)+1.361\cdot(T_4-T_3)+0.111=0$ ...(d)

m = 4, Node 4: $\quad 0.766\cdot(T_3-T_4)+1.266\cdot(T_5-T_4)+0.111=0$ ...(e)

m = 5, Node 5: $\quad 0.81\cdot(T_4-T_5)+1.21\cdot(T_6-T_5)+0.111=0$ ...(f)

m = 6, Node 6: $\quad 0.84\cdot(T_5-T_6)+1.174\cdot(T_7-T_6)+0.111=0$ ...(g)

m = 7, Node 7: $\quad 0.862\cdot(T_6-T_7)+1.148\cdot(T_8-T_7)+0.111=0$ ...(h)

m = 8, Node 8: $\quad 0.879\cdot(T_7-T_8)+1.129\cdot(T_9-T_8)+0.11=0$ ...(i)

m = 9, Node 9: $\quad 0.892\cdot(T_8-T_9)+1.114\cdot(T_{10}-T_9)+0.11=0$ ...(j)

For centre node '0': Apply Eq. 8.37:

$$6\cdot(T_1-T_0)+\frac{(\Delta r)^2\cdot q_0}{k}=0 \quad ...(8.37)$$

i.e. $\quad 6\cdot(T_1-T_0)+0.111=0$ ...(a)

For boundary node 10: Since there is convection condition at the surface, Eq. 8.39 can be applied. However, in this problem, since all the heat generated in the sphere has to be dissipated at the surface by convection, we can make an energy balance at the surface and get the temperature at node 10 directly:

Making an energy balance at the surface of the sphere,

$$\frac{4}{3}\cdot\pi\cdot R^3\cdot q_g=h_a\cdot(4\cdot\pi\cdot R^2)\cdot(T_{10}-T_a)$$

i.e.

$$T_{10}:=T_a+\frac{\frac{4}{3}\cdot\pi\cdot R^3\cdot q_g}{h_a\cdot(4\cdot\pi\cdot R^2)} \quad \textit{(define } T_{10}\textit{)}$$

$$T_{10}=23.333°\text{C} \quad ...(k)$$

So, now we have 11 Eqs. a to k for 11 node temperatures T_0 to T_{10} and by solving these equations simultaneously, we get the temperatures at the different nodes.

We use 'solve block' of Mathcad to solve this set of equations. Start with guess values for all unknown temperatures and immediately below 'Given', type the constraint equations. Then, the command 'Find (T_0,...T_{10})' gives the temperatures immediately:

$T_0 := 50 \quad T_2 := 50 \quad T_3 := 50 \quad T_4 := 50$

$T_5 := 50 \quad T_6 := 50 \quad T_7 := 50 \quad T_8 := 50 \quad T_9 := 50 \quad T_{10} := 23.333$ *(guess values of temperatures)*

Given

$6\cdot(T_1-T_0)+0.111=0$ ...(a)

$0.25\cdot(T_0-T_1)+2.25\cdot(T_2-T_1)+0.111=0$ ...(b)

$0.563\cdot(T_1-T_2)+1.563\cdot(T_3-T_2)+0.111=0$ ...(c)

$0.694\cdot(T_2-T_3)+1.361\cdot(T_4-T_3)+0.111=0$ ...(d)

$$0.766 \cdot (T_3 - T_4) + 1.266 \cdot (T_5 - T_4) + 0.111 = 0 \quad \text{...(e)}$$
$$0.81 \cdot (T_4 - T_5) + 1.21 \cdot (T_6 - T_5) + 0.111 = 0 \quad \text{...(f)}$$
$$0.84 \cdot (T_5 - T_6) + 1.174 \cdot (T_7 - T_6) + 0.111 = 0 \quad \text{...(g)}$$
$$0.862 \cdot (T_6 - T_7) + 1.148 \cdot (T_8 - T_7) + 0.111 = 0 \quad \text{...(h)}$$
$$0.879 \cdot (T_7 - T_8) + 1.129 \cdot (T_9 - T_8) + 0.11 = 0 \quad \text{...(i)}$$
$$0.892 \cdot (T_8 - T_9) + 1.114 \cdot (T_{10} - T_9) + 0.11 = 0 \quad \text{...(j)}$$
$$T_{10} = 23.333°C \quad \text{...(k)}$$

Temp: = Find($T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}$)

i.e. Temp =

	0
0	25.163
1	25.144
2	25.093
3	25.003
4	24.876
5	24.712
6	24.51
7	24.271
8	23.994
9	23.682
10	23.333

i.e. the node temperatures are:	corresponding radius at the nodes
$T_0 := 25.163°C$	$rad_0 := 0.0$ m
$T_1 := 25.144°C$	$rad_1 := 0.001$ m
$T_2 := 25.093°C$	$rad_2 := 0.002$ m
$T_3 := 25.003°C$	$rad_3 := 0.003$ m
$T_4 := 24.876°C$	$rad_4 := 0.004$ m
$T_5 := 24.712°C$	$rad_5 := 0.005$ m
$T_6 := 24.51°C$	$rad_6 := 0.006$ m
$T_7 := 24.271°C$	$rad_7 := 0.007$ m
$T_8 := 23.994°C$	$rad_8 := 0.008$ m
$T_9 := 23.682°C$	$rad_9 := 0.009$ m
$T_{10} := 23.333°C$	$rad_{10} := 0.01$ m

To compare numerical results with analytical temperature distribution:

For a sphere with internal heat generation, analytical expression for temperature distribution is:

$$\text{Temp}(r) := T_a + \frac{q_g \cdot R}{3 \cdot h_a} + \frac{q_g}{6 \cdot k} \cdot (R^2 - r^2) \quad \text{...(5.42)}$$

Substitute different values for r and get temperatures at corresponding nodes:

	Analytical solution	Numerical soluion with 11 nodes
Node 0:	$T(0.0) = 25.185$	$T_0 = 25.163°C$
Node 1:	$T(0.001) = 25.167$	$T_1 = 25.144°C$
Node 2:	$T(0.002) = 25.111$	$T_2 = 25.093°C$
Node 3:	$T(0.003) = 25.019$	$T_3 = 25.003°C$
Node 4:	$T(0.004) = 24.889$	$T_4 = 24.876°C$
Node 5:	$T(0.005) = 24.722$	$T_5 = 24.712°C$
Node 6:	$T(0.006) = 24.519$	$T_6 = 24.51°C$
Node 7:	$T(0.007) = 24.278$	$T_7 = 24.271°C$
Node 8:	$T(0.008) = 24$	$T_8 = 23.994°C$
Node 9:	$T(0.009) = 23.685$	$T_9 = 23.682°C$
Node 10:	$T(0.01) = 23.333$	$T_{10} = 23.333°C$

To sketch the temperature profile in the sphere, define a range variable r, varying from 0 to 0.01 m, with an increment of 0.0005 m. Then, choose x-y graph from the graph palette, and fill up the place holders on the x-axis and y-axis with r and Temp(r) respectively. Click anywhere outside the graph region, and immediately the graph appears.

Also, to sketch the temperature profile from numerical results in the same graph for comparison, define a range variable i = 0 to 10 (nodes) and on the x-axis, next to r put a comma and fill up rad_i and on the y-axis fill up T_i after putting a comma next to Temp(r), as shown.

Click anywhere outside the graph and both the graphs appear immediately: See Fig. Ex. 8.5(b)

$r := 0, 0.0005, ..., 0.01$ *(define a range variable r..starting value = 0, next value = 0.0005 m and last value = 0.01 m)*

$i := 0, ..., 10$ *(define a range variable, i varying from 0 to 10, with an increment of 1)*

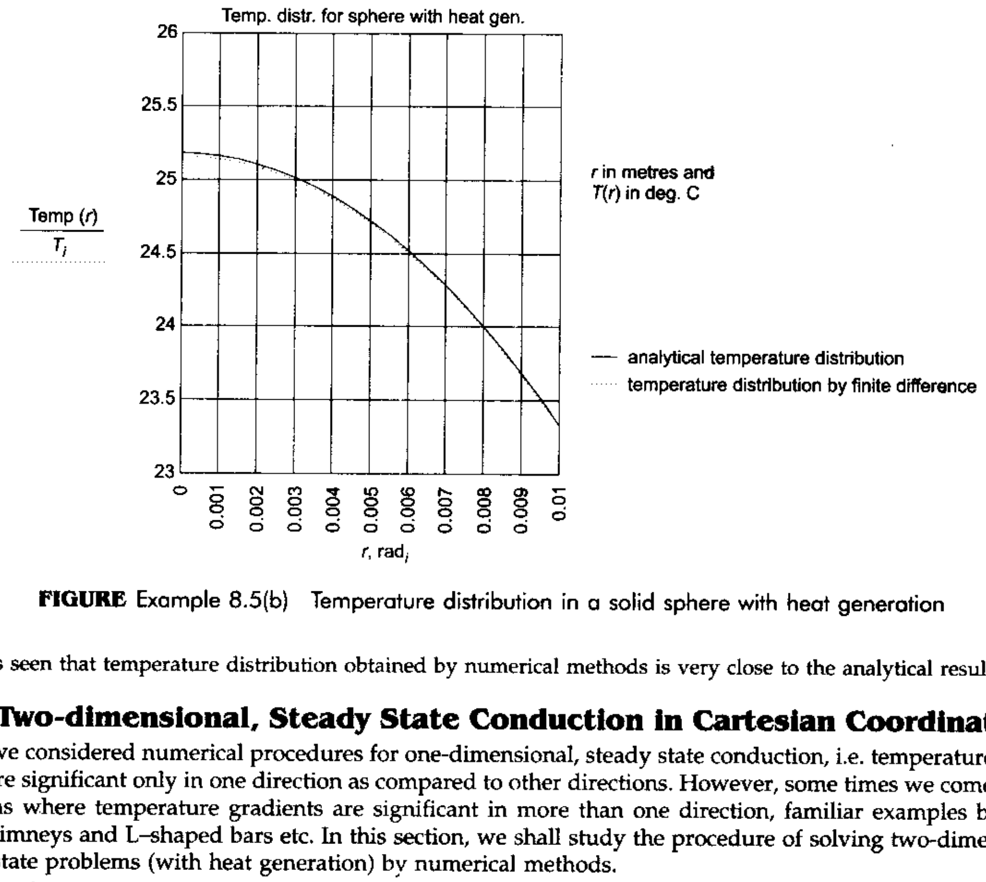


FIGURE Example 8.5(b) Temperature distribution in a solid sphere with heat generation

It is seen that temperature distribution obtained by numerical methods is very close to the analytical results.

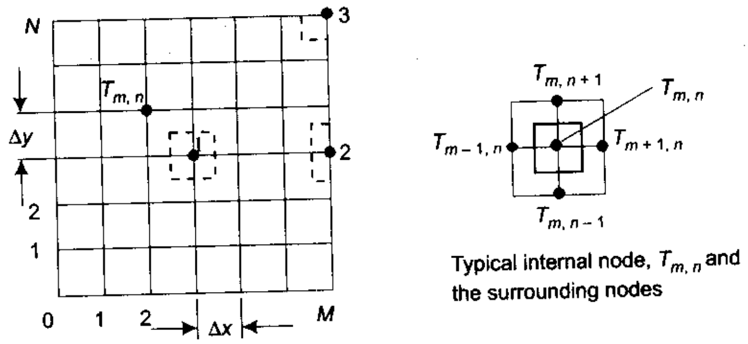
8.7 Two-dimensional, Steady State Conduction in Cartesian Coordinates

So far, we considered numerical procedures for one-dimensional, steady state conduction, i.e. temperature gradients were significant only in one direction as compared to other directions. However, some times we come across problems where temperature gradients are significant in more than one direction, familiar examples being in large chimneys and L–shaped bars etc. In this section, we shall study the procedure of solving two-dimensional steady state problems (with heat generation) by numerical methods.

Consider a two-dimensional system in which temperature gradients are significant in the x and y directions. Let the x–y plane be subdivided into rectangular mesh of nodes, with spacing of Δx and Δy in x and y directions respectively. Then, the nodes are numbered with a double subscript notation. i.e. a typical node, $T_{m,n}$ is the node with a x-coordinate of $(m.\Delta x)$ and y-coordinate of $(n.\Delta y)$. Node count is $m = 0, 1, ..., M$ in the x direction and $n = 0, 1, ..., N$ in the y direction. See Fig. 8.6.

We see that there are basically three types of nodes: internal nodes, surface nodes, and corner nodes, marked 1, 2 and 3 respectively in the Fig. 8.6 (a).

Difference equations for different nodes are written in the usual manner by making an energy balance for the elemental volume around the node in question, with all the heat flow lines going into the volume. Elemental volumes for the internal node, surface node and corner nodes are shown by dotted lines around the nodes, in the Fig. 8.6.



(a) (b)

FIGURE 8.6 Finite difference representation for two-dimensional conduction-nodal network

Difference equations for internal nodes:

Consider a typical internal node, $T_{m,n}$ in the x–y plane, with unit depth perpendicular to the plane of paper, as shown in Fig. 8.6 (b). It is surrounded by 4 nodes: $T_{m-1,n}$, $T_{m,n+1}$, $T_{m+1,n}$, and $T_{m,n-1}$. Let us make an energy balance on the elemental volume surrounding the node $T_{m,n}$. It is observed that heat flows into the node from all the four directions, i.e. left, right, up and down. In addition, let there be heat generation in the volume at a rate of $(\Delta V.q_g)$, W, where q_g, (W/m^3), is the uniform volumetric heat generation rate in the system.

Writing the energy balance, in steady state,

$$Q_{left} + Q_{right} + Q_{up} + Q_{down} + \Delta V\cdot q_g = 0 \qquad ...(8.40)$$

i.e.

$$k\cdot\Delta y\cdot\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\cdot\Delta y\cdot\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k\cdot\Delta x\cdot\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$+ k\cdot\Delta x\cdot\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + q_g\cdot\Delta x\cdot\Delta y = 0$$

Simplifying, we get,

$$\frac{T_{m-1,n} - 2\cdot T_{m,n}\ T_{m+1,n}}{(\Delta x)^2} + \frac{T_{m,n-1} - 2\cdot T_{m,n} + T_{m,n+1}}{(\Delta y)^2} + \frac{q_g}{k} = 0 \qquad ...(8.41)$$

Eq. 8.41 gives the difference equation for internal nodes, i.e. for $m = 1, 2, ..., (M - 1)$, and $n = 1, 2, ..., (N - 1)$.

Now, generally a square mesh is used i.e. $\Delta x = \Delta y = (\Delta x$, say). Then, the Eq. 8.41 simplifies to:

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4\cdot T_{m,n} + \frac{q_g\cdot(\Delta x)^2}{k} = 0 \qquad ...(8.42)$$

Eq. 8.42 is the finite difference equation for the internal nodes, with $\Delta x = \Delta y$.

Note that the first 4 terms in Eq. 8.42 are the temperatures of the surrounding 4 nodes, and the last term is the heat generation term.

When there is no heat generation in the body, the difference equation for the node reduces to:

$$T_{m,n} = \frac{T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1}}{4} \qquad ...(8.43)$$

i.e. when there is no heat generation, and a square mesh is used in the analysis, temperature of an internal node is given as the arithmetic average of the surrounding four temperatures. ***It will be useful to remember this.***

Difference equations for boundary nodes:

Boundary nodes may be on the surface or on the corners. Difference equations are developed for boundary nodes in a similar manner as for interior nodes, i.e. by making an energy balance on the elemental volume surrounding the node. In Fig. 8.6 (a), we can see that the surface node 2 is surrounded by a half-volume and the corner node 3 has a quarter volume attached to it. Exact form of the difference equation will depend upon the boundary conditions i.e. prescribed temperature, prescribed heat flux, insulated, convection or radiation boundary conditions.

Fig. 8.7 shows some common boundary conditions encountered in practice.

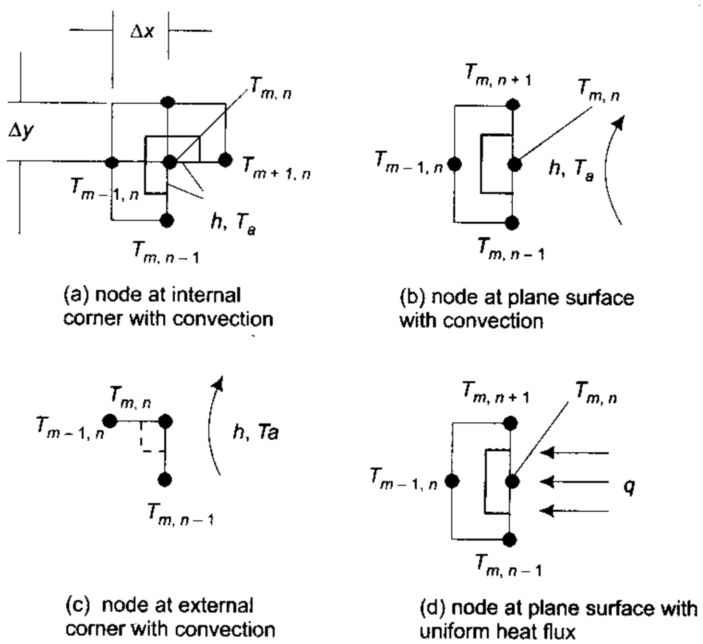


FIGURE 8.7 Finite difference representation for two-dimensional conduction-different boundary conditions

Example 8.6. Develop finite difference equations for an interior corner node with convection conditions, using the energy balance method. See Fig. 8.7 (a).

Solution. As shown in the Fig. 8.7, elemental volume around the node is ¾ of full volume. Writing an energy balance for this volume, we apply Eq. 8.40:

$$Q_{\text{left}} + Q_{\text{right}} + Q_{\text{up}} + Q_{\text{down}} + \Delta V \cdot q_g = 0 \qquad \text{...(8.40)}$$

i.e.

$$k \cdot \Delta y \cdot \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \cdot \frac{\Delta y}{2} \cdot \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k \cdot \Delta x \cdot \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$+ k \cdot \frac{\Delta x}{2} \cdot \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h \cdot \left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right) \cdot (T_a - T_{m,n}) + q_g \cdot \left(\frac{3}{4} \cdot \Delta x \cdot \Delta y\right) = 0$$

Remembering that $\Delta x = \Delta y = (\Delta x$, say), we get on simplification,

$$T_{m,n-1} + 2 \cdot T_{m-1,n} + 2 \cdot T_{m,n+1} + T_{m+1,n} - \left(6 + \frac{2 \cdot h \cdot \Delta x}{k}\right) \cdot T_{m,n} + \frac{3}{2} \cdot (\Delta x)^2 \cdot \frac{q_g}{k} + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a = 0$$

And, if there is no internal heat generation,

$$T_{m,n-1} + 2 \cdot T_{m-1,n} + 2 \cdot T_{m,n+1} + T_{m+1,n} - \left(6 + \frac{2 \cdot h \cdot \Delta x}{k}\right) \cdot T_{m,n} + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a = 0 \qquad \text{...(8.44)}$$

Finite difference equations for the boundary conditions shown in Fig. 8.7 are summarized in Table 8.1.

Note: In Eqs. 8.42 and 8.44, put $h = 0$ or $q = 0$, to get difference equations for an insulated surface or a surface with thermal symmetry.

TABLE 8.1 Summary of steady state, finite difference equations for different boundary conditions (q = heat flux, h = convection heat transfer coefficient k = thermal conductivity, no internal heat generation, and $\Delta x = \Delta y$)

Situation	*Finite difference equation (with $\Delta x = \Delta y$, no heat generation)*
(1) Node at an internal corner with convection, Fig. 8.7,a:	
	$T_{m,n-1} + 2\cdot T_{m-1,n} + 2\cdot T_{m,n+1} + T_{m+1,n} - \left(6+\frac{2\cdot h\cdot \Delta x}{k}\right)\cdot T_{m,n} + \frac{2\cdot h\cdot \Delta x}{k}\cdot T_a = 0$ ...(8.44)
(2) Node at a plane surface with convection, Fig. 8.7,b:	
	$2\cdot T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{2\cdot h\cdot \Delta x}{k}\cdot T_a - 2\cdot\left(\frac{h\cdot \Delta x}{k}+2\right)\cdot T_{m,n} = 0$ ...(8.45)
(3) Node at an external corner with convection, Fig. 8.7,c:	
	$(T_{m,n-1} + T_{m-1,n}) + \frac{2\cdot h\cdot \Delta x}{k}\cdot T_a - 2\cdot\left(\frac{h\cdot \Delta x}{k}+1\right)\cdot T_{m,n} = 0$ ...(8.46)
(4) Node at a plane surface with uniform heat flux, Fig. 8.7,d:	
	$(2\cdot T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2\cdot q\cdot \Delta x}{k} - 4\cdot T_{m,n} = 0$ ...(8.47)

Example 8.7. Develop difference equation for a tool tip, shown in Fig. Example 8.7. There is uniform heat flux, q (W/m^2) on the upper surface. Assume a constant thickness, t for the tool tip.

Solution. Note that the node (m, n) is enclosed by surfaces AB, BC and CA.

Finite difference equation is developed by writing the energy balance on the elemental volume surrounding node (m, n) i.e. on the volume enclosed by AB, BC and CA, remembering that we assume all heat flow lines into the volume element.. We get:

$$\frac{\Delta x}{2}\cdot t\cdot q + k\cdot\frac{\Delta y}{2}\cdot t\cdot\frac{T_{m+1,n}-T_{m,n}}{\Delta x} + h\cdot t\cdot\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{2}\cdot(T_a - T_{m,n}) = 0$$

In the above equation, first term is the heat flux through surface AB, second term is the conduction across surface BC, and third term is convection at surface CA.

Simplifying the above equation, with $\Delta x = \Delta y$, we get:

$$\left(\frac{k}{\Delta x}\right)\cdot T_{m+1,n} - \left(\frac{k}{\Delta x}+\sqrt{2}\cdot h\right)\cdot T_{m,n} + (q+\sqrt{2}\cdot h\cdot T_a) = 0$$

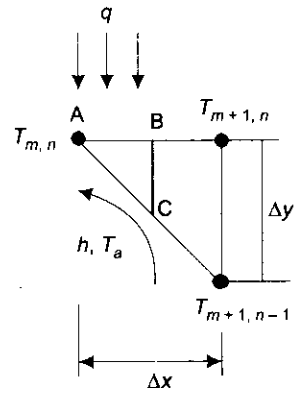


FIGURE Example 8.7 Finite difference representation for a tool tip

Above equation gives the desired difference equation for the tool tip.

Example 8.8. For the two-dimensional region shown in Fig. Example 8.8, with constant k (= 20 W/(mC)) and no internal heat generation, and with the indicated boundary conditions, formulate the finite difference equations and solve for unknown temperatures. Use $\Delta x = \Delta y$ = 1 cm.

Solution.

Data:

Δx :=0.01 m Δy := 0.01 m T_a := 20°C h := 50 W/(m^{2}C) k := 20 W/(mC)

Nodes are represented by numbers 1, 2, ..., 7. Elemental volume pertinent to each node is also marked around it and numbered a, b, ..., r.

We shall develop finite difference equation for each node by writing the energy balance for the corresponding elemental volume around that node.

For node 1:

Elemental volume to be considered is 1/4 volume, 1-a-b-c-1.

For this elemental volume, considering unit depth, heat transfers into the volume are:

From left surface, there is no heat transfer, since it is insulated.

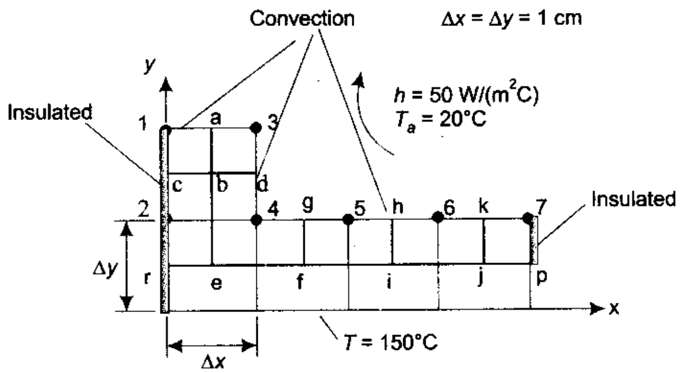


FIGURE Example 8.8 Finite difference representation for two-dimensional conduction-nodal network

i.e. $Q_{left} = 0$

Form top, i.e surface 1-a: there is convection:

i.e. $$Q_{top} = h \cdot \left(\frac{\Delta x}{2} \cdot 1\right) \cdot (T_a - T_1)$$

From right, there is conduction from node 3 through surface a-b:

i.e. $$Q_{right} = k \cdot \left(\frac{\Delta y}{2} \cdot 1\right) \cdot \frac{T_3 - T_1}{\Delta x}$$

From down, there is conduction from node 2 through surface b-c:

i.e. $$Q_{down} = k \cdot \left(\frac{\Delta x}{2} \cdot 1\right) \cdot \frac{T_2 - T_1}{\Delta y}$$

There is no heat generation term in this problem.

So, heat balance on the elemental volume for node 1 gives:

$$h \cdot \left(\frac{\Delta x}{2} \cdot 1\right) \cdot (T_a - T_1) + k \cdot \left(\frac{\Delta y}{2} \cdot 1\right) \cdot \frac{T_3 - T_1}{\Delta x} + k \cdot \left(\frac{\Delta x}{2} \cdot 1\right) \cdot \frac{T_2 - T_1}{\Delta y} = 0$$

i.e. $$0.25 \cdot (T_a - T_1) + 10 \cdot (T_3 - T_1) + 10 \cdot (T_2 - T_1) = 0 \quad \text{...(a)}$$

Eq. a is the difference equation for node 1.

For node 2:

Here, elemental volume to be considered is 1/2 volume, c-b-e-r. and, energy balance can be written as we did for node 1.

However, since the surface is insulated, it is easier to use the mirror image concept and consider the node 2 as an internal node. So, to the left of node 2, we have T_4, mirror image of temperature of node 4. Then, considering 2 as internal node, we get difference equation for node 2:

$$T_4 + T_1 + T_4 + 150 - 4 \cdot T_2 = 0 \quad \text{...(b)}$$

For node 3:

This is a corner node with convection. Elemental volume to be considered is 1/4 volume, a-3-d-b.

We can directly apply Eq. 8.46, viz.

$$(T_{m,\,n-1} + T_{m-1,\,n}) + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a - 2 \cdot \left(\frac{h \cdot \Delta x}{k} + 1\right) \cdot T_{m,\,n} = 0 \quad \text{...(8.46)}$$

i.e. $$T_4 + T_1 + 0.05 \cdot T_a - 2.05 \cdot T_3 = 0 \quad \text{...(c)}$$

For node 4:

This is an internal corner node with convection. Elemental volume to be considered is 3/4 volume, g-f-e-b-d-4.

Again, we can directly apply Eq. 8.44, viz.

$$T_{m,\,n-1} + 2 \cdot T_{m-1,\,n} + 2 \cdot T_{m,\,n+1} + T_{m+1,\,n} - \left(6 + \frac{2 \cdot h \cdot \Delta x}{k}\right) \cdot T_{m,\,n} + \frac{2 \cdot h \cdot \Delta x}{k} \cdot T_a = 0 \quad \text{...(8.44)}$$

i.e. $$150 + 2\cdot T_2 + 2\cdot T_3 + T_5 - 6.05\cdot T_4 + 0.05\cdot T_a = 0 \quad \text{...(d)}$$

Eq. d is the difference equation for node 4.

For node 5:

This is a surface node with convection. Elemental volume to be considered is 1/2 volume, g-f-i-h-g.

Again, we can directly apply Eq. 8.45, viz.

$$2\cdot T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{2\cdot h\cdot \Delta x}{k}\cdot T_a - 2\cdot\left(\frac{h\cdot\Delta x}{k} + 2\right)\cdot T_{m,n} = 0 \quad \text{...(8.45)}$$

Remembering that Eq. 8.45 was developed for a vertical surface, and in the present case, we are dealing with a horizontal surface, we can write:

$$2\cdot 150 + T_4 + T_6 + 0.05\cdot T_a - 4.05\cdot T_5 = 0 \quad \text{...(e)}$$

For node 6:

This is identical to node 5. So, we get:

$$2\cdot 150 + T_5 + T_7 + 0.05\cdot T_a - 4.05\cdot T_6 = 0 \quad \text{...(f)}$$

For node 7:

This is a corner node with conduction from left, convection on the top, insulated on the right, and conduction from down. Elemental volume to be considered is 1/4 volume, k-7-p-j.

Writing the energy balance:

$$k\cdot\left(\frac{\Delta y}{2}\cdot 1\right)\cdot\frac{T_6 - T_7}{\Delta x} + h\cdot\left(\frac{\Delta x}{2}\cdot 1\right)\cdot(T_a - T_7) + 0 + k\cdot\left(\frac{\Delta x}{2}\cdot 1\right)\cdot\frac{150 - T_7}{\Delta y} = 0$$

i.e. $$10\cdot(T_6 - T_7) + 0.25\cdot(T_a - T_7) + 10\cdot(150 - T_7) = 0 \quad \text{...(g)}$$

Temperatures at nodes 1 to 7 are obtained by simultaneously solving 7 Eqs. a to g.

We use 'solve block' of Mathcad to solve this set of equations. Start with guess values for all unknown temperatures and immediately below 'Given', type the constraint equations. Then, the command 'Find (T_1, ..., T_7)' gives the temperatures immediately:

$T_1 := 50$ $T_2 := 50$ $T_3 := 50$ $T_4 := 50$ $T_5 := 50$ $T_6 := 50$ $T_7 := 50$ (guess values of temperatures)

Given

$$0.25\cdot(T_a - T_1) + 10\cdot(T_3 - T_1) + 10\cdot(T_2 - T_1) = 0 \quad \text{...(a)}$$
$$T_4 + T_1 + T_4 + 150 - 4\cdot T_2 = 0 \quad \text{...(b)}$$
$$T_4 + T_1 + 0.05\cdot T_a - 2.05\cdot T_3 = 0 \quad \text{...(c)}$$
$$150 + 2\cdot T_2 + 2\cdot T_3 + T_5 - 6.05\cdot T_4 + 0.05\cdot T_a = 0 \quad \text{...(d)}$$
$$2.150 + T_4 + T_6 + 0.05\cdot T_a - 4.05\cdot T_5 = 0 \quad \text{...(e)}$$
$$2.150 + T_5 + T_7 + 0.05\cdot T_a - 4.05\cdot T_6 = 0 \quad \text{...(f)}$$
$$10\cdot(T_6 - T_7) + 0.25\cdot(T_a - T_7) + 10\cdot(150 - T_7) = 0 \quad \text{...(g)}$$

$\text{Temp} := \text{Find}(T_1, T_2, T_3, T_4, T_5, T_6, T_7)$ (*node temperatures are stored in vector 'Temp'.*)

i.e. $$\text{Temp} = \begin{bmatrix} 138.552 \\ 142.929 \\ 137.139 \\ 141.582 \\ 145.437 \\ 146.438 \\ 146.636 \end{bmatrix}$$

i.e. The node temperatures are:

$T_1 = 138.552°C$ $T_2 = 142.929°C$ $T_3 = 137.139°C$ $T_4 = 141.582°C$

$T_5 = 145.437°C$ $T_6 = 146.438°C$ $T_7 = 146.636°C$

Note: In Mathcad, while using the solve block, a great advantage is that the equations can be written in any order; also, there is no need to collect the coefficient of each variable separately. Equations can be entered without simplification, as in the form we get after making the energy balance.

Example 8.9. A very long bar of square cross-section has its four sides held at constant temperatures as shown in Fig. Example 8.9. Determine the temperatures at the internal nodes. Compare the results with analytical solution.

Solution. There are 9 internal nodes. Difference equations for these nodes are obtained by applying Eq. 8.42, viz.

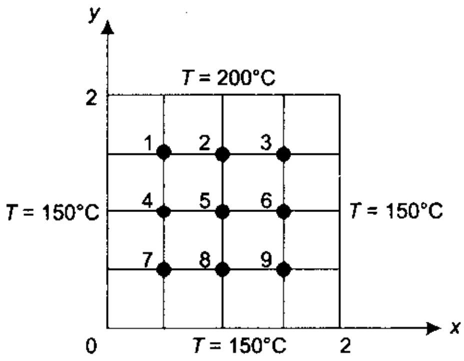


FIGURE Example 8.9 Finite difference representation for two-dimensional conduction-nodal network

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4 \cdot T_{m,n} + \frac{q_g \cdot (\Delta x)^2}{k} = 0 \qquad ...(8.42)$$

In the present case, there is no internal heat generation. So, the last term of the above equation will be zero. Therefore, we get:

Node 1:	$150 + 200 + T_2 + T_4 - 4 \cdot T_1 = 0$	...(a)
Node 2:	$T_1 + 200 + T_3 + T_5 - 4 \cdot T_2 = 0$	...(b)
Node 3:	$T_2 + 200 + 150 + T_6 - 4 \cdot T_3 = 0$	...(c)
Node 4:	$150 + T_1 + T_5 + T_7 - 4 \cdot T_4 = 0$	...(d)
Node 5:	$T_4 + T_2 + T_6 + T_8 - 4 \cdot T_5 = 0$	...(e)
Node 6:	$T_5 + T_3 + 150 + T_9 - 4 \cdot T_6 = 0$	...(f)
Node 7:	$150 + T_4 + T_8 + 150 - 4 \cdot T_7 = 0$	...(g)
Node 8:	$T_7 + T_5 + T_9 + 150 - 4 \cdot T_8 = 0$	...(h)
Node 9:	$T_8 + T_6 + 150 + 150 - 4 \cdot T_9 = 0$	...(i)

Eqs. a to i give 9 difference equations for 9 internal nodes. By solving these equations simultaneously, we get the temperatures at nodes 1 to 9.

We use 'solve block' of Mathcad to solve this set of equations. Start with guess values for all unknown temperatures and immediately below 'Given', type the constraint equations. Then, the command 'Find (T_1, ..., T_9)' gives the temperatures immediately:

$T_1 := 50 \quad T_2 := 50 \quad T_3 := 50 \quad T_4 := 50 \quad T_5 := 50$

$T_6 := 50 \quad T_7 := 50 \quad T_8 := 50 \quad T_9 := 50$ *(guess values of temperatures)*

Given

$$150 + 200 + T_2 + T_4 - 4 \cdot T_1 = 0 \qquad ...(a)$$
$$T_1 + 200 + T_3 + T_5 - 4 \cdot T_2 = 0 \qquad ...(b)$$
$$T_2 + 200 + 150 + T_6 - 4 \cdot T_3 = 0 \qquad ...(c)$$
$$150 + T_1 + T_5 + T_7 - 4 \cdot T_4 = 0 \qquad ...(d)$$
$$T_4 + T_2 + T_6 + T_8 - 4 \cdot T_5 = 0 \qquad ...(e)$$
$$T_5 + T_3 + 150 + T_9 - 4 \cdot T_6 = 0 \qquad ...(f)$$
$$150 + T_4 + T_8 + 150 - 4 \cdot T_7 = 0 \qquad ...(g)$$
$$T_7 + T_5 + T_9 + 150 - 4 \cdot T_8 = 0 \qquad ...(h)$$
$$T_8 + T_6 + 150 + 150 - 4 \cdot T_9 = 0 \qquad ...(i)$$

Temp := Find($T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9$) *(node temperatures are stored in vector 'Temp'.)*

i.e.
$$\text{Temp} = \begin{bmatrix} 171.429 \\ 176.339 \\ 171.429 \\ 159.375 \\ 162.5 \\ 159.375 \\ 153.571 \\ 154.911 \\ 153.571 \end{bmatrix}$$

i.e. The node temperatures are:

$T_1 = 171.429°C \quad T_2 = 176.339°C \quad T_3 = 171.429°C \quad T_4 = 159.375°C \quad T_5 = 162.5°C \quad T_6 = 159.375°C$

$T_7 = 153.571°C \quad T_8 = 154.911°C \quad T_9 = 153.571°C$

Comparison with analytical solution:

Analytical solution for this problem is a little complicated and is given in terms of an infinite series, as follows:

$$\theta = \theta_c \cdot \frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n} \cdot \frac{\sinh\left(n \cdot \pi \cdot \frac{y}{L}\right)}{\sinh\left(n \cdot \pi \cdot \frac{W}{L}\right)} \cdot \sin\left(n \cdot \pi \cdot \frac{x}{L}\right)$$

Nomenclature for the above equation for the present problem is as follows:

$\theta = T - 150$ *(T = temperature at the desired point; 150°C is the constant temperature on three sides)*

$\theta_c := 200 - 150$ *(temperature difference between the temperature of fourth side and the consant temperature)*

n *(number of terms considered in the infinite series)*

x, y *(coordinates of the point where temperature is desired)*

$L := 2$ m *(length along x-axis)*

$W := 2$ m *(length along y-axis)*

Above equation is solved very easily in Mathcad:

Let us re-define θ as a function of (x, y), and consider only 6 terms of the infinite series ($n = 6$) as shown below, for convenience:

$$\theta(x, y) := \theta_c \cdot \frac{2}{\pi} \cdot \sum_{n=1}^{6} \frac{(-1)^{n+1} + 1}{n} \cdot \frac{\sinh\left(n \cdot \pi \cdot \frac{y}{L}\right)}{\sinh\left(n \cdot \pi \cdot \frac{W}{L}\right)} \cdot \sin\left(n \cdot \pi \cdot \frac{x}{L}\right) \quad ...(A)$$ *(define θ as a function of x and y)*

Now, substitute (x, y) corresponding to different nodes and get the analytical temperature at those nodes immediately:

Node	$\theta(x, y)$	Temperature of node (deg. C)		Temperature by numerical method
1	$\theta(0.5, 1.5) = 21.623$	$T_1 = \theta(0.5, 1.5) + 150$	$T_1 = 171.623$	171.429
2	$\theta(1.0, 1.5) = 27.059$	$T_2 = \theta(1.0, 1.5) + 150$	$T_2 = 177.059$	176.339
3	$\theta(1.5, 1.5) = 21.623$	$T_3 = \theta(1.5, 1.5) + 150$	$T_3 = 171.623$	171.429
4	$\theta(0.5, 1.0) = 9.102$	$T_4 = \theta(0.5, 1.0) + 150$	$T_4 = 159.102$	159.375
5	$\theta(1.0, 1.0) = 12.5$	$T_5 = \theta(1.0, 1.0) + 150$	$T_5 = 162.5$	162.5
6	$\theta(1.5, 1.0) = 9.102$	$T_6 = \theta(1.5, 1.0) + 150$	$T_6 = 159.102$	159.375
7	$\theta(0.5, 0.5) = 3.399$	$T_7 = \theta(0.5, 0.5) + 150$	$T_7 = 153.399$	153.571
8	$\theta(1.0, 0.5) = 4.771$	$T_8 = \theta(1.0, 0.5) + 150$	$T_8 = 154.771$	154.991
9	$\theta(1.5, 0.5) = 3.399$	$T_9 = \theta(1.5, 0.5) + 150$	$T_9 = 153.399$	153.571

We make following important observations:

(i) Even with a crude mesh of 4 × 4, we get values of temperatures at the nodes very close to the analytical results.

(ii) Note that the analytical relation to find the temperature at any point is very complicated, and to solve it without a computer is rather laborious and time consuming. But, with Mathcad, even this analytical solution is easy to perform.

(iii) Numerical method of formulating difference equations by energy balance method is easy and straight forward, only labour being in solving the set of simultaneous equations. But, with Mathcad, this is also very easy.

Irregular boundaries:

Very often, in practice, we have to analyse bodies with irregular boundaries, e.g. engine blocks, turbine blades, aerofoil sections etc. In such cases, the whole volume cannot be entirely filled up by the square mesh used for numerical analysis. Still, easiest and simple approach to deal with such geometries is to fill the geometry by approximating the irregular boundary with simple square mesh elements, as shown in Fig. 8.8. This method, generally gives results of acceptable accuracy, particularly when the mesh size is small and the nodes are quite close to each other. Many commercially available specialized software for numerical analysis employ more sophisticated methods.

8.8 Numerical Methods for Transient Heat Conduction

In transient conduction, temperature varies with both position and time. So, to obtain finite difference equations for transient conduction, we have to discretize both space and time domains. This scheme is illustrated in Fig. 8.9.

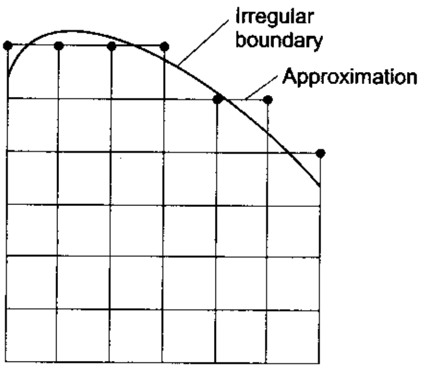


FIGURE 8.8 Approximating an irregular boundary

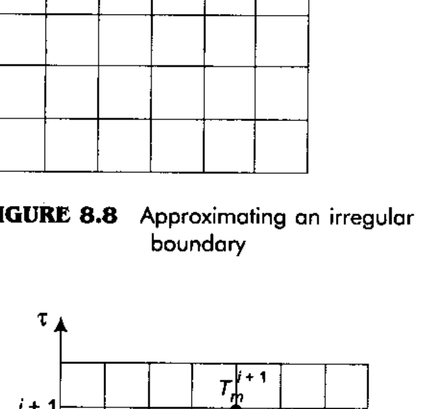


Figure 8.9 Finite difference representation for one-dimensional, transient conduction-nodal network

Here, the space increment is Δx and the time increment is $\Delta\tau$. At a given node 'm', x – coordinate is $(m.\Delta x)$ and at a given time step 'i', time from start up is $(i.\Delta\tau)$, as is clear from the Fig. 8.9. Starting from initial temperature at $\tau = 0$, at each node we calculate the temperature at a successive time interval of $\Delta\tau$ till we reach the desired time at which temperature has to be calculated. Therefore, obviously, number of calculations required in case of transient conduction is much more. Time step is shown in superscript, i.e. T_m^i is the temperature of node 'm' at time step 'i' (at time = $i.\Delta\tau$ from start up) and the notation T_m^{i+1} means the temperature of node 'm' at the time step $(i + 1)$ (at time = $(i + 1)\Delta\tau$ from start up).

Formulation of finite difference equations in transient conduction is done by an energy balance on the elemental volumes containing the nodes, just as was done in the case of steady state conduction; however, now, on the RHS, there appears a term representing the change in energy content of the elemental volume, with time. Also, as in the earlier case, while writing the energy balance, it is assumed that all heat lines flow *into* the elemental volume.

We write, for a given volume element:

(Heat transferred into the volume element from all sides, per unit time) + (Heat generated within the volume element per unit time) = (Change in energy content of the volume element per unit time).

$$(Q_{left} + Q_{up} + Q_{right} + Q_{down} + Q_g = \rho \cdot V_{element} \cdot C_p \cdot \left[\frac{T_m^{(i+1)} - T_m^i}{\Delta\tau}\right] \quad ...(8.48)$$

In the above equation, as already mentioned, T_m^i is the temperature of node 'm' at time step 'i' (i.e. at time = $i.\Delta\tau$ from start up) and T_m^{i+1} is the temperature of node 'm' at the time step $(i + 1)$ (i.e. at time = $(i + 1)\Delta\tau$ from start up). C_p is the specific heat and ρ is the density of the medium. $(T_m^{i+1} - T_m^i)/\Delta\tau$ is the finite difference approximation of the term $dT/d\tau$.

Now, regarding the terms on the LHS of Eq. 8.48, the question arises as to whether we should consider the temperatures of the nodes at step 'i' or step '$(i + 1)$'. In fact, both the methods are adopted in practice. While applying Eqs. 8.48 to write the finite difference equation for a node, if the terms on the LHS of the equation are considered at time step 'i', then, the method is known as explicit method of approach; if the terms on the LHS of the equation are considered at time step '$(i + 1)$', then, the method is known as implicit method of approach. To summarize:

Explicit method:

$$(Q_{left}^i + Q_{up}^i + Q_{right}^i + Q_{down}^i + Q_g^i) = \rho \cdot V_{element} \cdot C_p \cdot \left[\frac{T_m^{(i+1)} - T_m^i}{\Delta\tau}\right] \quad ...(8.49)$$

Implicit method:

$$(Q_{left}^{i+1} + Q_{up}^{i+1} + Q_{right}^{i+1} + Q_{down}^{i+1}) + Q_g^{i+1} = \rho \cdot V_{element} \cdot C_p \cdot \left[\frac{T_m^{(i+1)} - T_m^i}{\Delta\tau}\right] \quad ...(8.50)$$

In the explicit method, time derivative is calculated in 'forward difference' form, and in implicit method, the time derivative is in the 'backward difference' form.

Eqs. 8.49 and 8.50 are applicable in any coordinate system and for multidimensional systems too; however, when more than one-dimension is involved, number of surfaces through which heat flows into the elemental volume are more and there will be correspondingly more terms on the LHS of Eqs. 8.49 and 8.50.

Explicit method is called so, because temperature of the node 'm' at time step $(i + 1)$ is calculated explicitly in terms of the temperatures calculated at the previous time step 'i'; therefore, the calculations are quite straight forward; however it suffers from a serious limitation that the time increment cannot be independently fixed, but has an upper limit because of stability considerations. But in case of implicit method, this limitation on time duration is not there and we can choose any time step; but the implicit method requires that at each time step, nodal temperatures have to be solved simultaneously.

8.8.1 One-dimensional Transient Heat Conduction in a Plane Wall

Consider one-dimensional, transient heat conduction in a plane wall of thickness L, with heat generation rate $q_g(x, \tau)$ and constant thermal conductivity k. Now, let us divide the region $0 < x < L$ into M sub-regions. Then, thickness of each sub-region is:

$\Delta x = L/M$. So, there are totally $(M + 1)$ nodes, starting from $m = 0$ to $m = M$, as shown in Fig. 8.10. Coordinate of node 'm' is $x = m.\Delta x$. and let temperature of node 'm' be T_m. Remembering that each node represents the sub-volume around it (of thickness Δx), it is clear that interior nodes 1, 2...$M - 1$ represent full sub-volumes whereas boundary nodes 0 and M represent half volumes (of thickness $\Delta x/2$). Volume of element surrounding node 'm' is $A.\Delta x$.

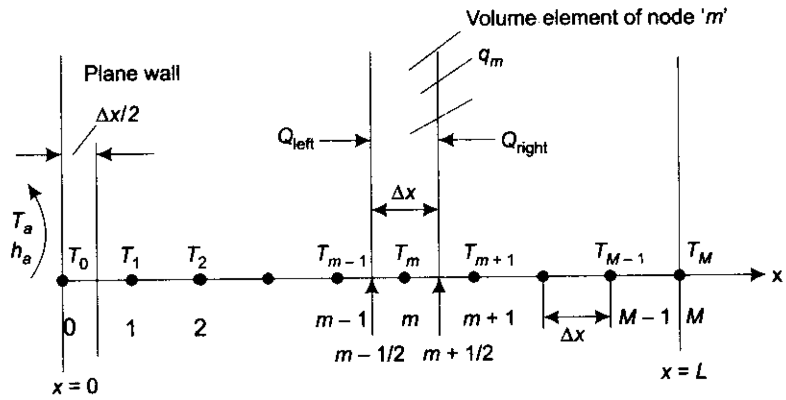


FIGURE 8.10 Finite difference formulation in a plane wall by energy balance for transient heat conduction

To get the finite difference formulation, we apply the general energy balance, i.e. Eq. 8.48:

$$k\cdot A\cdot\frac{T_{m-1}-T_m}{\Delta x}+k\cdot A\cdot\frac{T_{m+1}-T_m}{\Delta x}+q_m\cdot(A\cdot\Delta x)=\rho A\cdot\Delta x\cdot C_p\cdot\frac{T_m^{i+1}-T_m^i}{\Delta\tau} \quad ...(8.51)$$

Simplifying,

$$T_{m-1}-2\cdot T_m+T_{m+1}+\frac{q_m\cdot(\Delta x)^2}{k}=\frac{(\Delta x)^2}{\alpha\cdot\Delta\tau}\cdot(T_m^{i+1}-T_m^i) \quad ...(8.52)$$

where, $\alpha = \frac{k}{\rho\cdot C_p}$ = thermal diffisivity of the material.

Now, the term $\frac{\alpha\cdot\Delta\tau}{(\Delta x)^2}$ is the finite difference form of the Fourier number, Fo

So, Eq. 8.52 reduces to:

$$T_{m-1}-2\cdot T_m+T_{m+1}+\frac{q_m\cdot(\Delta x)^2}{k}=\frac{(T_m)^{i+1}-(T_m)^i}{Fo} \quad ...(8.53)$$

Now, as we mentioned earlier, in the LHS of Eq. 8.53, we can use the temperatures of the nodes at the 'previous time step, i', or temperatures at the 'next time step, i + 1'. If we use temperatures at time step 'i', it is the 'explicit method' and if the temperatures at time step 'i + 1' are used, then, it is the 'implicit method'.

Explicit method:

$$(T_{m-1})^i - 2\cdot(T_m)^i + (T_{m+1})^i + \frac{q_m^i\cdot(\Delta x)^2}{k} = \frac{(T_m)^{i+1} - (T_m)^i}{Fo} \quad ...(8.54)$$

Now, the new temperature T_m^{i+1} can be explicitly solved since the other terms involved at the previous time step 'i', are already known. So, we write for T_m^{i+1}:

$$(T_m)^{i+1} = Fo\cdot[(T_{m-1})^i + (T_{m+1})^i] + (1 - 2\cdot Fo)\cdot(T_m)^i + Fo\cdot\frac{(q_m)^i\cdot(\Delta x)^2}{k} \quad ...(8.55)$$

Eq. 8.55 is the explicit difference equation valid for all interior nodes, 1, 2, ..., $(M-1)$, when there is internal heat generation.

When there is no heat generation, Eq. 8.55 reduces to:

$$(T_m)^{i+1} = Fo\cdot[(T_{m-1})^i + (T_{m+1})^i] + (1 - 2\cdot Fo)\cdot(T_m)^i \quad ...(8.56)$$

Implicit method:

If in the LHS of Eq. 8.53, we use the values at time step $(i + 1)$, we get the implicit relation for the node temperatures:

i.e.
$$(T_{m-1})^{i+1} - 2\cdot(T_m)^{i+1} + (T_{m+1})^{i+1} + \frac{(q_m)^{i+1}\cdot(\Delta x)^2}{k} = \frac{(T_m)^{i+1} - (T_m)^i}{Fo} \quad ...(8.57)$$

Eq. 8.57 is simplified to:

$$(1+ 2\cdot Fo)\cdot(T_m)^{i+1} - F_o\left[(T_{m-1})^{i+1} + (T_{m+1})^{i+1} + \frac{(q_m)^{i+1}\cdot(\Delta x)^2}{k}\right] = +\ (T_m)^i \quad ...(8.58)$$

Eq. 8.58 is the implicit difference equation valid for all interior nodes, 1,2,...,$(M-1)$, when ther is internal heat generation.

When there is no heat generation, Eq. 8.58 reduces to:

$$(1 + 2\cdot Fo)\cdot(T_m)^{i+1} - Fo\cdot[(T_{m-1})^{i+1} + (T_{m+1})^{i+1}] - (T_m)^i = 0 \quad ...(8.59)$$

With the use of either the explicit or the implicit equations given above, we get $M - 1$ nodal equations. Unless the temperatures at the boundaries are specified in the problem, we need two more equations for the boundary nodes '0' and 'M'. These are obtained by applying the energy balance for the half-volumes around these nodes. See Fig. 8.10. Exact nature of the difference equations depends on the specific boundary condition. For example:

For node '0' with convection boundary condition:

Explicit formulation:

$$h\cdot A\cdot[T_a - (T_0)^i] + k\cdot A\cdot\frac{(T_1)^i - (T_0)^i}{\Delta x} + (q_0)^i\cdot A\cdot\frac{\Delta x}{2} = \rho\cdot A\cdot\frac{\Delta x}{2}\cdot C_p\cdot\frac{(T_0)^{i+1} - (T_0)^i}{\Delta\tau} \quad ...(8.60)$$

Simplifying:

$$(T_0)^{i+1} = (1 - 2\cdot Fo - 2\cdot Fo\cdot Bi)\cdot(T_0)^i + Fo\cdot\left[2\cdot(T_1)^i + 2\cdot Bi\cdot T_a + \frac{(q_0)^i\cdot(\Delta x)^2}{k}\right] \quad ...(8.61)$$

where $Bi = \dfrac{h\cdot\Delta x}{k}$ = Biot number

When there is no heat generation, Eq. 8.61 for explicit formulation becomes:

$$(T_0)^{i+1} = (1 - 2\cdot Fo - 2\cdot Fo\cdot Bi)\cdot(T_0)^i + Fo\cdot[2\cdot(T_1)^i + 2\cdot Bi\cdot T_a] \quad ...(8.62)$$

For other types of boundary conditions, difference equations are developed in a similar manner, by applying the energy balance on the elemental volume containing the node and considering all the heat flows to be *into* the volume.

Once the difference equations are developed for all nodes with suitable Δx, next step is to choose a suitable time increment $\Delta\tau$. Then, starting with the initial conditions at $\tau = 0$, solve the difference equations for the temperatures T_m^{i+1} at all the nodes at the next time step $\tau = \Delta\tau$. Now, using these values of temperatures as 'previous values', again get the nodal temperatures at the next time step $\tau = 2.\Delta\tau$, using the same difference equations. Thus, continue to *march in time* till the solution is obtained for the desired time interval.

Stability criterion:

We said that once the explicit difference equations are developed, suitable time interval $\Delta\tau$ has to be chosen. This is done keeping the stability criterion in mind, since the explicit method is not unconditionally stable. That means, above a certain value of $\Delta\tau$, the solution will not converge. This limit on $\Delta\tau$ is determined from mathematical and thermodynamic considerations (see good text books on numerical methods) as follows:

"Coefficients of all T_m^i in the T_m^{i+1} expressions (called 'primary coefficients') must be greater than or equal to zero for all nodes 'm'".

Considering Eq. 8.55 for interior nodes, we see that coefficient of T_m^i is $(1 - 2.Fo)$ and applying the above mentioned criterion for stability, we get:

$$1 - 2 \cdot Fo \geq 0$$

i.e. $$Fo = \frac{\alpha \cdot \Delta\tau}{(\Delta x)^2} \leq \frac{1}{2}$$ (*for interior nodes, one-dimensional conduction*...(8.63))

Now, $\Delta\tau$ must be fixed from Eq. 8.63.

However, generally, boundary nodes with convection conditions are more restrictive and in such cases, coefficient of T_m^i from the most restrictive eqn. must be considered for the stability criterion and the time step $\Delta\tau$ must be determined with respect to that coefficient.

Example 8.10. A large uranium plate of thickness $L = 10$ cm, ($k = 28$ W/(mC), $\alpha = 12.5 \times 10^{-6}$ m^2/s) is initially at an uniform temperature of 100°C. Heat generation rate in the plate is 5×10^6 W/m^3. At time $\tau = 0$, both the left and right sides of the plate are subjected to convection with a fluid at at temperature of 0°C and a heat transfer coefficient of 1500 W/(m^{2}C). Using a uniform nodal spacing of 2 cm, develop the explicit finite difference formulations for all nodes, and determine the temperature distribution in the plate after 5 min. Also, find out how long it will take for steady conditions to be reached in the plate.

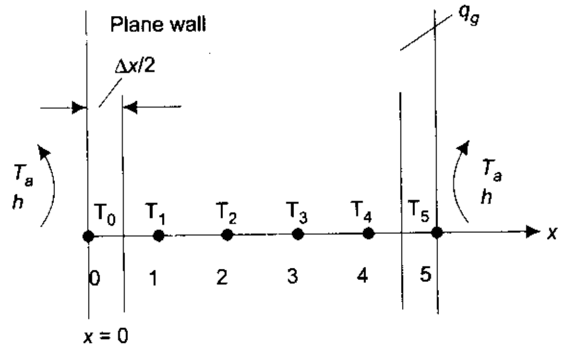


FIGURE Example 8.10 Finite difference formulation in a plate by energy balance for transient heat conduction

(b) Also, solve this problem by implicit finite difference formulation.

Solution.

Data:

$L := 0.1$ m $k := 28$ W/(mC) $\alpha := 12.5 \times 10^{-6}$ m^2/s $q_g := 5 \times 10^6$ W/m^3 $T := 100$°C $T_a := 0$°C

$\Delta x := 0.02$ m $h := 1500$ W/(m^{2}C) (*convective heat transfer coeff.*) $M := 5$ $\tau := 300$ s

Difference equations for interior nodes:

Nodes 1, 2, 3 and 4 are interior nodes. Finite difference equations for these nodes by explicit method are obtained from Eq. 8.55, by setting $m = 1, 2, 3, 4$. i.e.

$$(T_m)^{i+1} = Fo \cdot [(T_{m-1})^i + (T_{m+1})^i] + (1 - 2 \cdot Fo) \cdot (T_m)^i + Fo \cdot \frac{(q_m)^i \cdot (\Delta x)^2}{k} \quad ...(8.55)$$

We get:

Node 1: $$T_1^{i+1} = Fo[(T_0^i + T_2^i) + (1 - 2 \cdot Fo) \cdot T_1^i + Fo \cdot \frac{(q_g)^i \cdot (\Delta x)^2}{k} \quad ...(b)$$

Node 2: $$T_2^{i+1} = Fo[(T_1^i + T_3^i) + (1 - 2 \cdot Fo) \cdot T_2^i + Fo \cdot \frac{(q_g)^i \cdot (\Delta x)^2}{k} \quad ...(c)$$

Node 3: $$T_3^{i+1} = Fo[(T_2^i + T_4^i) + (1 - 2 \cdot Fo) \cdot T_3^i + Fo \cdot \frac{(q_g)^i \cdot (\Delta x)^2}{k} \quad ...(d)$$

Node 4: $$T_4^{i+1} = Fo[(T_3^i + T_5^i) + (1 - 2 \cdot Fo) \cdot T_4^i + Fo \cdot \frac{(q_g)^i \cdot (\Delta x)^2}{k} \quad ...(e)$$

Difference equation for boundary nodes:

For node '0':

Node '0' is on the left surface, subjected to convection. Applying the Eq. 8.61 directly:

$$(T_0)^{i+1} = (1 - 2\cdot Fo - 2\cdot Fo\cdot Bi)\cdot(T_0)^i + Fo\cdot\left[2\cdot(T_1)^i + 2\cdot Bi\cdot T_a + \frac{(q_0)^i\cdot(\Delta x)^2}{k}\right] \quad ...(8.61)$$

where $B_i = \frac{h\cdot\Delta x}{k}$ = Biot number

i.e.

$$(T_0)^{i+1} = (1 - 2\cdot Fo - 2\cdot Fo\cdot Bi)\cdot(T_0)^i + Fo\cdot\left[2\cdot(T_1)^i + 2\cdot Bi\cdot T_a + \frac{(q_g)\cdot(\Delta x)^2}{k}\right] \quad ...(a)$$

For node 5:

This is a node with convection boundary condition. So, applying the energy balance to the half-volume around node 5, with all the heat lines flowing into the element, we get:

$$h\cdot A\cdot(T_a - T_5^i) + k\cdot A\cdot\left(\frac{T_4^i - T_5^i}{\Delta x}\right) + q_g\cdot A\cdot\frac{\Delta x}{2} = \rho\cdot A\cdot\frac{\Delta x}{2}\cdot C_p\cdot\frac{T_5^{i+1} - T_5^i}{\Delta\tau}$$

i.e.

$$T_5^{i+1} = (1 - 2\cdot F_o - 2\cdot F_o\cdot B_i)\cdot(T_5)^i + F_o\cdot\left[2\cdot(T_4)^i + 2\cdot Bi\cdot T_a + \frac{(q_a)\cdot(\Delta x)^2}{k}\right] \quad ...(f)$$

Now, we have to fix the upper limit of $\Delta\tau$ from stability criterion. To do that, we observe that in Eqs. a to f, the smaller coefficient of T_m^i is in Eq. f, i.e. (1 – 2. Fo –2.$Fo.B_i$) must be greater than or equal to zero. Putting this condition, we get:

$$1 - 2\cdot Fo - 2\cdot Fo\cdot\frac{h\cdot\Delta x}{k} \geq 0$$

i.e.

$$Fo \leq \frac{1}{2\cdot\left(1 + \frac{h\cdot\Delta x}{k}\right)}$$

i.e.

$$\Delta\tau \leq \frac{(\Delta x)^2}{2\cdot\alpha\cdot\left(1 + \frac{h\cdot\Delta x}{k}\right)}$$

i.e. $\Delta\tau \leq 7.724$ s

This means that a time step less than 7.724 s has to be employed from stability criterion.

Let us choose:

$\Delta\tau := 5$ s

Then, $Fo := \frac{\alpha\cdot\Delta\tau}{(\Delta x)^2}$ i.e. $Fo = 0.1563$

Substituting all relevant numerical values in Eqs. a to f, we get the explicit difference equations as:

$$T_0^{i+1} = 0.353\cdot T_0^i + 0.1563\cdot(2\cdot T_1^i + 71.429) \quad ...(a)$$
$$T_1^{i+1} = 0.1563\cdot(T_0^i + T_2^i) + 0.688\cdot T_1^i + 11.161 \quad ...(b)$$
$$T_2^{i+1} = 0.1563\cdot(T_1^i + T_3^i) + 0.688\cdot T_2^i + 11.161 \quad ...(c)$$
$$T_3^{i+1} = 0.1563\cdot(T_2^i + T_4^i) + 0.688\cdot T_3^i + 11.161 \quad ...(d)$$
$$T_4^{i+1} = 0.1563\cdot(T_3^i + T_5^i) + 0.688\cdot T_4^i + 11.161 \quad ...(e)$$
$$T_5^{i+1} = 0.363\cdot(T_5)^i + 0.1563\cdot[2\cdot(T_4)^i + 71.429] \quad ...(f)$$

Initial temperature of the plate at $\tau = 0$ and $i = 0$, is given as 100°C.

i.e. $T_0^0 = T_1^0 = T_3^0 = T_4^0 = T_5^0 = 100$°C

Therefore, at the next time step $i = 1$, i.e. at $\Delta\tau = 5$ s, temperatures at nodes 0 to 5 can be explicitly calculated from Eqs. a to f. Then, calculate temperatures at the nodes for next time step of $\Delta\tau = 10$ s, using the same Eqs. a to f, since the temperatures at the previous time step are already calculated. Thus, march in time till we reach the time limit specified in the problem, 5 min, i.e. there are 60 time steps of 5 s each.

This calculation is easily done in Mathcad. We slightly change the notation for convenience in calculation: we write the superscripts as subscripts to work with matrix notation, as shown below.

In the small Mathcad program given below, LHS defines a function 'Temp(n)' where n is the no. of time steps, which we can specify. Output is a vector containing step no., total time elapsed, and node temperatures $T_0, T_1, ..., T_5$.

• On the RHS, first 6 lines define the initial temperatures at the nodes, all equal to 100°C.

Then, a 'for loop' evaluates the finite difference Eqs. a to f, each 'new' node temperature being calculated in terms of the temperatures calculated in the previous time step. Here, the number of time steps, 'n' can be changed since it is included in function definition on the LHS.

$$
\text{Temp}(n) = \left\| \begin{array}{l}
T0_0 \leftarrow 100 \\
T1_0 \leftarrow 100 \\
T2_0 \leftarrow 100 \\
T3_0 \leftarrow 100 \\
T4_0 \leftarrow 100 \\
T5_0 \leftarrow 100 \\
\text{for } i \in 0..\, n \\
\quad \left\| \begin{array}{l}
T0_{i+1} \leftarrow 0.353 \cdot T0_i + 0.1563 \cdot (2 \cdot T1_i + 71.429) \\
T1_{i+1} \leftarrow 0.1563 \cdot (T0_i + T2_i) + 0.688 \cdot T1_i + 11.161 \\
T2_{i+1} \leftarrow 0.1563 \cdot (T1_i + T3_i) + 0.688 \cdot T2_i + 11.161 \\
T3_{i+1} \leftarrow 0.1563 \cdot (T2_i + T4_i) + 0.688 \cdot T3_i + 11.161 \\
T4_{i+1} \leftarrow 0.1563 \cdot (T3_i + T5_i) + 0.688 \cdot T4_i + 11.161 \\
T5_{i+1} \leftarrow 0.353 \cdot T5_i + 0.1563 \cdot (2 \cdot T4_i + 71.429)
\end{array} \right. \\
[i \;\; 5 \cdot i \;\; T0_i \;\; T1_i \;\; T2_i \;\; T3_i \;\; T4_i \;\; T5_i]
\end{array} \right.
$$

Temp(0) = [0 0 100 100 100 100 100 100] *(starting at time = 0)*

i = step no.; $\Delta\tau$= one time step = 5 s; τ = time duration from beginning = $i.\ \Delta\tau$, s

	i	τ	T_0	T_1	T_2	T_3	T_4	T_5
Temp(2) =	[2	10	73.369	117.213	122.449	122.449	117.213	73.369]
Temp(4) =	[4	20	75.447	127.666	142.472	142.472	127.666	75.447]
Temp(12) =	[12	60	95.576	169.805	202.468	202.468	169.805	95.576]
Temp(18) =	[18	90	108.991	196.256	236.479	236.479	196.256	108.991]
Temp(24) =	[24	120	120.036	217.958	264.179	264.179	217.958	120.036]
Temp(36) =	[36	180	136.45	250.194	305.285	305.285	250.194	136.45]
Temp(48) =	[48	240	147.409	271.714	332.724	332.724	271.714	145.409]
Temp(60) =	[60	300	154.724	286.079	351.041	351.041	286.079	154.724]
Temp(120) =	[120	600	167.466	311.102	382.942	382.946	311.102	167.466]
Temp(180) =	[180	900	169.155	314.419	387.176	387.176	314.419	169.155]
Temp(250) =	[250	1.25×10^3	169.388	314.878	387.761	387.761	314.878	169.388]
Temp(260) =	[260	1.3×10^3	169.395	314.891	387.778	387.778	314.891	169.395]

Temperature distribution after 5 min.:

Above Table of results gives node temperatures at different time steps.

Temp(60) corresponds to 60th time step, i.e. 300 s from beginning.

We note that after 5 min. the node temperatures are:

$T_0 = T_5 = 154.724°C$; $T_1 = T_4 = 286.07°C$; $T_2 = T_3 = 351.041°C$;

Time to reach steady state:

It may be seen from the Table that from about 240th step, the temperatures at the nodes do not vary much as we advance in time, i.e. steady state is reached at about 20 min. from start up.

To draw the temperatures at the nodes at different times:

It is instructive to graphically represent the manner in which the plate proceeds to attain steady state temperature. First represent the node temperatures at different time steps as vectors:

$$Step0 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$ *(intial temperature distribution in nodes* 0, 1, ..., 5)

Similarly, temperature distributions after 1, 5, 10, 20 and 30 min. are given Step1, Step5,...,etc., below:

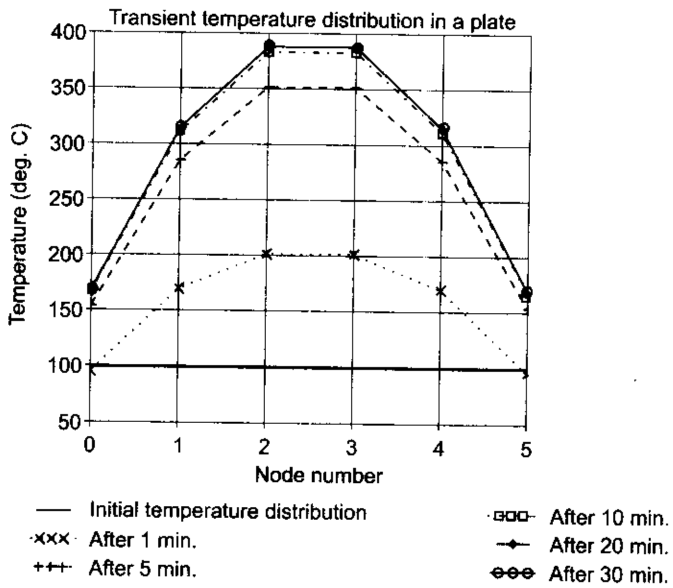
$$Step1 = \begin{bmatrix} 95.58 \\ 169.81 \\ 202.47 \\ 202.47 \\ 169.81 \\ 95.58 \end{bmatrix} \quad Step5 = \begin{bmatrix} 154.72 \\ 286.08 \\ 351.04 \\ 351.04 \\ 286.08 \\ 154.72 \end{bmatrix} \quad Step10 = \begin{bmatrix} 167.47 \\ 311.1 \\ 382.95 \\ 382.95 \\ 311.1 \\ 167.47 \end{bmatrix}$$

$$Step20 = \begin{bmatrix} 169.38 \\ 314.86 \\ 387.74 \\ 387.74 \\ 314.86 \\ 169.38 \end{bmatrix} \quad Step30 = \begin{bmatrix} 169.41 \\ 314.92 \\ 387.82 \\ 387.82 \\ 314.92 \\ 169.41 \end{bmatrix}$$

To draw the graph, first define a range variable i = 0 to 5 with an increment of 1. This represents nodes 0, 1, ..., 5. Select the x–y graph from the graph palette and fill up 'i' in the place holder on the x-axis. In the place holder of y-axis, fill up above shown temperature vectors, with a comma between each. Click anywhere outside the graph region and the graphs appear:

i := 0, ..., 5 *(define a range variable 'i' varying from* 0 *to* 5, *with an increment of* 1)

It is seen from the graph that steady state is reached at about 20 min. from start up.



(b) Implicit method:

Difference equations for interior nodes:

Nodes 1, 2, 3 and 4 are interior nodes. Finite difference equations for these nodes by implicit method are obtained from Eq. 8.58, by setting m = 1, 2, 3, 4. i.e.

$$(1 + 2\cdot Fo)\cdot(T_m)^{i+1} - Fo\cdot\left[(T_{m-1})^{i+1} + (T_{m+1})^{i+1} + \frac{(q_m)^{i+1}\cdot(\Delta x)^2}{k}\right] - (T_m)^i = 0 \qquad ...(8.58)$$

Node 1:
$$(1 + 2\cdot Fo)\cdot T_1^{i+1} - Fo\cdot\left[(T_0)^{i+1} + (T_2)^{i+1} + \frac{(q_g)\cdot(\Delta x)^2}{k}\right] - T_1^i = 0 \qquad ...(b)$$

Node 2:
$$(1 + 2\cdot Fo)\cdot T_2^{i+1} - Fo\cdot\left[(T_1)^{i+1} + (T_3)^{i+1} + \frac{(q_g)\cdot(\Delta x)^2}{k}\right] - T_2^i = 0 \qquad ...(c)$$

Node 3:
$$(1 + 2\cdot Fo)\cdot T_3^{i+1} - Fo\cdot\left[(T_2)^{i+1} + (T_4)^{i+1} + \frac{(q_g)\cdot(\Delta x)^2}{k}\right] - T_3^i = 0 \qquad ...(d)$$

Node 4:
$$(1 + 2\cdot Fo)\cdot T_4^{i+1} - Fo\cdot\left[(T_3)^{i+1} + (T_5)^{i+1} + \frac{(q_g)\cdot(\Delta x)^2}{k}\right] - T_4^i = 0 \qquad ...(e)$$

Difference equations for boundary nodes:

Nodes 0 and 5 are boundary nodes, with convection conditions.

For node '0':

Writing the energy balance for the half-volume around node '0', with all heat flow lines going into the volume element, with the LHS of Eq. 8.60 expressed at time step (i + 1), we get:

$$h\cdot A\cdot[T_a - (T_0)^{i+1}] + k\cdot A\cdot\frac{(T_1)^{i+1} - (T_0)^{i+1}}{\Delta x} + q_g\cdot A\cdot\frac{\Delta x}{2} = \rho\cdot A\cdot\frac{\Delta x}{2}\cdot C_p\cdot\frac{(T_0)^{i+1} - (T_0)^i}{\Delta\tau}$$

i.e.
$$\frac{2\cdot Fo\cdot h\cdot\Delta x}{k}\cdot[T_a - (T_0)^{i+1}] + 2\cdot Fo\cdot[(T_1)^{i+1} - (T_0)^{i+1}] + \frac{Fo\cdot q_g\cdot(\Delta x)^2}{k} = (T_0)^{i+1} - (T_0)^i \qquad ...(a)$$

Eq. a is the implicit finite difference formulation for node '0', with convection conditions.

For node '5':

Writing the energy balance for the half-volume around node '5', with all heat flow lines going into the volume element, with the LHS of energy balance equation expressed at time step (i + 1), we get:

$$h\cdot A\cdot(T_a - T_5^{i+1}) + k\cdot A\cdot\left(\frac{T_4^{i+1} - T_5^{i+1}}{\Delta x}\right) + q_g\cdot A\cdot\frac{\Delta x}{2} = \rho\cdot A\cdot\frac{\Delta x}{2}\cdot C_p\cdot\frac{T_5^{i+1} - T_5^i}{\Delta\tau}$$

i.e.
$$\frac{2\cdot Fo\cdot h\cdot\Delta x}{k}\cdot[T_a - (T_5)^{i-1}] + 2\cdot Fo\cdot[(T_4)^{i+1} - (T_5)^{i+1}] + \frac{Fo\cdot q_g\cdot(\Delta x)^2}{k} = (T_5)^{i+1} - (T_5)^i \qquad ...(f)$$

Eq. f is the implicit finite difference formulation for node 5, with convection conditions.

Now, we can choose any $\Delta\tau$, since there is no problem of stability in implicit formulation.

Let us choose:

$$\Delta\tau := 10 \text{ s}$$

Therefore,
$$Fo := \frac{\alpha\cdot\Delta\tau}{(\Delta x)^2} \quad \text{i.e.} \quad Fo = 0.3125$$

Inserting numerical values, Eqs. a to f are written as:

$$0.67\cdot[T_a - (T_0)^{i+1}] + 0.625\cdot[(T_1)^{i+1} - (T_0)^{i-1}] + 22.321 = (T_0)^{i+1} - (T_0)^i \qquad ...(a)$$
$$1.625\cdot T_1^{i+1} - 0.3125\cdot[(T_0)^{i+1} - (T_2)^{i+1} + 71.429] - T_1^i = 0 \qquad ...(b)$$
$$1.625\cdot T_2^{i+1} - 0.3125\cdot[(T_1)^{i+1} + (T_3)^{i+1} + 71.429] - T_2^i = 0 \qquad ...(c)$$
$$1.625\cdot T_3^{i+1} - 0.3125\cdot[(T_2)^{i+1} + (T_4)^{i+1} + 71.429] - T_3^i = 0 \qquad ...(d)$$
$$1.625\cdot T_4^{i+1} - 0.3125\cdot[(T_3)^{i+1} + (T_5)^{i+1} + 71.429] - T_4^i = 0 \qquad ...(e)$$
$$0.67\cdot[T_a - (T_5)^{i+1}] + 0.625\cdot[(T_4)^{i+1} + (T_5)^{i+1}] + 22.321 = (T_5)^{i+1} - (T_5)^i \qquad ...(f)$$

Now, to start with, i.e. at $\tau = 0$, all the node temperatures $T_0, T_1, ..., T_5$ are known. Then, at the next time step, solve Eqs. a to f simultaneously to get the node temperatures at that time step. Using these results, solve the Eqs. a to f at the next time step, etc. till you reach the given time limit.

This part of the problem is left as an exercise for the student. Write a computer program to accomplish this task. Use the Gauss–Siedel iteration technique for the solution of simultaneous equations.

8.8.2 Two-dimensional Transient Heat Conduction

Fig. 8.11 shows a rectangular region where the heat transfer in x and y directions are significant, and heat transfer in the z direction is negligible. Divide the rectangular region into a nodal network of thicknesses Δx and Δy as shown. Let the thickness in the z direction be unity.

Finite difference equations are developed by writing the energy balance for an elemental volume surrounding the node under consideration. All heat flows are considered to be flowing *into* the volume.

Difference equations for interior nodes:

A typical interior node, $T_{m,n}$ and the elemental volume surrounding it, and immediate neighbours of this node are shown in Fig. 8.11 (b). Node $T_{m,n}$ is surrounded by 4 nodes: $T_{m-1,n}$, $T_{m,n+1}$, $T_{m+1,n}$, and $T_{m,n-1}$. Let us make an energy balance on the elemental volume surrounding the node $T_{m,n}$. It is observed that heat flows into the node from all the four directions, i.e. left, up, right and down. In addition, let there be heat generation in the volume at a rate of $(\Delta V.q_g)$, W, where q_g, (W/m^3), is the uniform heat generation rate in the system.

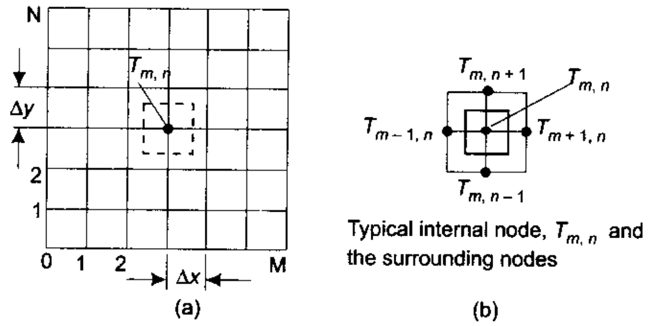


FIGURE 8.11 Finite difference representation for two-dimensional conduction-nodal network

Writing the energy balance,

$$Q_{left} + Q_{up} + Q_{right} + Q_{down} + \Delta V \cdot q_g = m \cdot C_p \cdot \frac{dT}{d\tau} \qquad ...(8.64)$$

i.e.

$$k \cdot \Delta y \cdot \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \cdot \Delta x \cdot \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k \cdot \Delta y \cdot \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$+ k \cdot \Delta x \cdot \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + q_g \cdot \Delta x \cdot \Delta y = \rho \cdot \Delta x \cdot \Delta y \cdot C_p \cdot \frac{T_m^{i+1} - T_m^i}{\Delta \tau} \qquad ...(8.65)$$

For $\Delta x = \Delta y$ (i.e. a square mesh), we get:

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4 \cdot T_{m,n} + \frac{q_g \cdot (\Delta x)^2}{k} = \frac{T_m^{i+1} - T_m^i}{Fo} \qquad ...(8.66)$$

where $Fo = \frac{\alpha \cdot \Delta \tau}{(\Delta x)^2}$ = Fourier number, and α is thermal diffusivity.

Now, on the LHS of Eq. 8.66, if we use the 'previous' time step 'i', we get the explicit formulation of finite difference equation for interior nodes:

i.e.

$$(T_{m-1,n})^i + (T_{m+1,n})^i + (T_{m,n+1})^i + (T_{m,n-1})^i - (4\cdot T_{m,n})^i + \frac{q_g\cdot(\Delta x)^2}{k} = \frac{(T_{m,n})^{i+1} - (T_{m,n})^i}{Fo} \quad ...(8.67)$$

i.e.

$$(T_{m,n})^{i+1} = Fo\cdot[(T_{m-1,n})^i + (T_{m+1,n})^i + (T_{m,n+1})^i + (T_{m,n-1})^i] + (1 - 4\cdot Fo)\cdot(T_{m,n})^i + Fo\cdot\frac{q_g\cdot(\Delta x)^2}{k} \quad ...(8.68)$$

Eq. 8.68 is valid for all interior nodes, when there is heat generation.

If there is no heat generation, Eq. 8.68 simplifies to:

$$(T_{m,n})^{i+1} = Fo\cdot[(T_{m-1,n})^i + (T_{m+1,n})^i + (T_{m,n+1})^i + (T_{m,n-1})^i] + (1 - 4\cdot Fo)\cdot(T_{m,n})^i \quad ...(8.69, a)$$

As mentioned earlier, stability criterion in the explicit method requires the coefficient of $(T_{m,n})^i$ to be positive and this condition gives the upper limit on the time increment $\Delta\tau$, as follows:

$$Fo = \frac{\alpha\cdot\Delta\tau}{(\Delta x)^2} \le \frac{1}{4} \quad \textit{(stability criterion for interior nodes...(8.70))}$$

Now, on the LHS of Eq. 8.66, if we use the 'future' time step '$i + 1$', we get the implicit formulation of finite difference equation for interior nodes. So, we get:

$$(T_{m,n})^{i+1} = (1 + 4\cdot Fo)\cdot T_{m,n}^{i+1} - Fo(T_{m+1,n}^{i+1} + T_{m-1,n}^{i+1} + T_{m,n+1}^{i+1} + T_{m,n-1}^{i+1}) \quad ...(8.69, b)$$

Difference equations for boundary nodes:

Boundary nodes may be on the surface or on the corners. Difference equations are developed for boundary nodes in a similar manner as for interior nodes, i.e. by making a heat balance on the elemental volume surrounding the node. Exact form of the difference equation will depend upon the boundary conditions i.e. prescribed temperature, prescribed heat flux, insulated, convection or radiation boundary conditions.

Fig. 8.12 shows some common boundary conditions encountered in practice:

Finite difference equations for the boundary situations shown in Fig. 8.12 are given in Table 8.2.

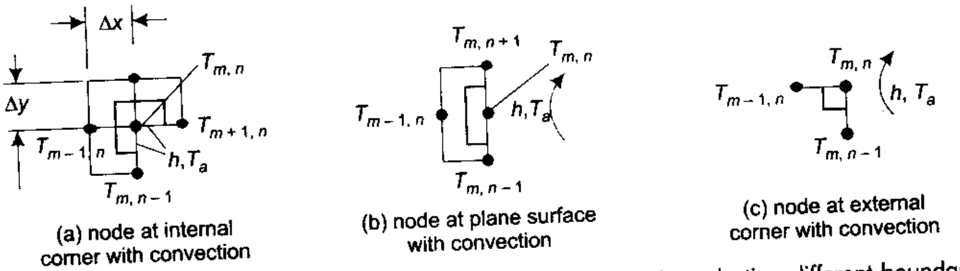


FIGURE 8.12 Finite difference representation for two-dimensional conduction-different boundary conditions

Example 8.11. Consider the L-bar shown in Fig. Example 8.11, with constant k (= 20 W/(mC)) and no internal heat generation. Its left and right sides are insulated and the bottom surface is maintained at 150°C at all times. If at time τ = 0, the top surface is suddenly exposed to a fluid at 20°C with a convection coefficient of 50 W/(m²C), determine the temperature at the node 3 after 1, 2, 5, 10, 15 and 20 min. Use explicit formulation and take $\Delta x = \Delta y = 1$ cm. Take thermal diffusivity of the body as 3.2×10^{-6} m²/s.

Solution.

Data:

$\Delta x := 0.01$ m $\Delta y := 0.01$ m $T_a := 20$°C $h := 50$ W/(m²C) $k := 20$ W/(mC) $\alpha := 3.2 \times 10^{-6}$ m²/s

Nodes are represented by numbers 1, 2, ..., 7. Elemental volume pertinent to each node is also marked around it and numbered $a, b, ..., r$.

We shall develop finite difference equations for each node by writing the energy balance for the corresponding elemental volume around that node.

TABLE 8.2 Summary of transient, finite difference equations for different boundary conditions (q = heat flux, h = convection heat transfer coefficient k = thermal conductivity, no internal heat generation, and $\Delta x = \Delta y$)

Situation	*Finite difference equation (with $\Delta x = \Delta y$, no heat generation)*
(1) Node at an internal corner with convection, Fig. 8.12,a:	
Explicit method:	$(T_{m,n})^{i+1} = \frac{2}{3}\cdot Fo\cdot[2\cdot(T_{m,n+1})^i + 2\cdot(T_{m-1,n})^i + (T_{m+1,n})^i + (T_{m,n-1})^i + 2\cdot Bi\cdot T_a] + \left(1 - 4\cdot Fo - \frac{4}{3}\cdot Fo\cdot Bi\right)\cdot(T_{m,n})^i$ ...(8.71)
Stability criterion for above:	$Fo\cdot(3 + Bi) \le \frac{3}{4}$ ...(8.72)
Implicit method:	$\left[1 + 4\cdot Fo\cdot\left(1 + \frac{Bi}{3}\right)\right]\cdot(T_{m,n})^{i+1} - \frac{2\cdot Fo}{3}\cdot[(T_{m+1,n})^{i+1} + (T_{m,n-1})^{i+1} + 2\cdot(T_{m,n-1})^{i+1} + 2\cdot(T_{m+1,n})^{i+1}] - \frac{4}{3}\cdot Bi\cdot Fo\cdot T_a - (T_{m,n})^i = 0$ ...(8.73)
2. Node at a plane surface with convection Fig. 8.12b:	
Explicit method:	$(T_{m,n})^{i+1} = Fo\cdot[2\cdot(T_{m-1,n})^i + (T_{m,n+1})^i + (T_{m,n-1})^i + 2\cdot Bi\cdot T_a] + (1 - 4\cdot Fo - 2\cdot Fo\cdot Bi)\cdot(T_{m,n})^i$ ...(8.74)
Stability criterion for above:	$Fo\cdot(2 + Bi) \le \frac{1}{2}$ ...(8.75)
Implicit method:	$[1 + 2\cdot Fo\cdot(2 + Bi)]^2\cdot(T_{m,n})^{i+1} - Fo\cdot[2\cdot(T_{m-1,n})^{i+1} + (T_{m,n+1})^{i+1} + (T_{m,n-1})^{i+1}] = (T_{m,n})^i + 2\cdot Bi\cdot Fo\cdot T_a$ ...(8.76)
3. Node at a plane surface, insulated:	
To obtain finite difference equation or stability criterion for an insulated surface (or a surface of thermal symmetry), set $B_i = 0$ (i.e. $h = 0$) in Eqs. 8.74, 8.75 or 8.76.	
4. Node at exterior corner, with convection Fig. 8.12c:	
Explicit method:	$(T_{m,n})^{i+1} = 2\cdot Fo\cdot[(T_{m-1,n})^i + (T_{m,n-1})^i + 2\cdot Bi\cdot T_a] + (1 - 4\cdot Fo - 4\cdot Fo\cdot Bi)\cdot(T_{m,n})^i$ ...(8.77)
Stability criterion for above:	$Fo\cdot(1 + Bi) \le \frac{1}{4}$ ...(8.78)
Implicit method:	$(1 + 4\cdot Fo\cdot(1 + Bi))\cdot(T_{m,n})^{i+1} - 2\cdot Fo\cdot[(T_{m-1,n})^{i+1} + (T_{m,n-1})^{i+1}] = (T_{m,n})^i + 4\cdot Bi\cdot Fo\cdot T_a$ ...(8.79)

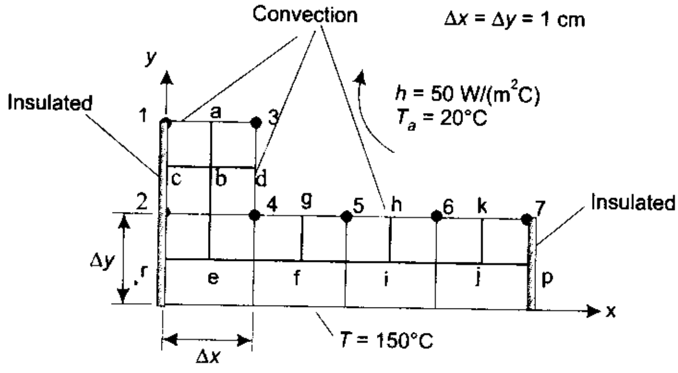


FIGURE Example 8.11 Finite difference representation for two- dimensional conduction-nodal network

For node 1:
Elemental volume to be considered is 1/4 volume, 1-a-b-c-1. This node is subjected to convection from top and conduction from right and conduction from bottom.

From left surface, there is no heat transfer, since it is insulated.

Considering all heat flows to be flowing into the elemental volume, and writing an energy balance, we get:

$$h\cdot\left(\frac{\Delta x}{2}\cdot 1\right)\cdot(T_a - T_1^i) + k\cdot\left(\frac{\Delta y}{2}\cdot 1\right)\cdot\frac{T_3^i - T_1^i}{\Delta x} + k\cdot\left(\frac{\Delta x}{2}\cdot 1\right)\cdot\frac{T_2^i - T_1^i}{\Delta y} = \rho\cdot\frac{\Delta x}{2}\cdot\frac{\Delta y}{2}\cdot C_p\cdot\frac{T_1^{i+1} - T_1^i}{\Delta\tau}$$

Remembering that $\Delta x = \Delta y$, and dividing by $k/4$ and simplifying:

$$\frac{2\cdot h\cdot\Delta x}{k}\cdot(T_a - T_1^i) + 2\cdot(T_3^i - T_1^i) + 2\cdot(T_2^i - T_1^i) = \frac{T_1^{i+1} - T_1^i}{Fo}$$

i.e.

$$T_1^{i+1} = \left(1 - 4\cdot Fo - 2\cdot Fo\cdot\frac{h\cdot\Delta x}{k}\right)\cdot T_1^i + 2\cdot Fo\cdot\left(T_3^i + T_2^i + \frac{h\cdot\Delta x}{k}\cdot T_a\right) \quad \text{...(a)}$$

Eq. a is the explicit difference equation for temperature of node 1 at $(i + 1)$st time step.

For node 2:
Here, elemental volume to be considered is 1/2 volume, c-b-e-r. and, energy balance can be written as we did for node 1.

However, since the surface is insulated, it is easier to use the mirror image concept and consider the node 2 as an internal node. So, to the left of node 2, we have T_4, mirror image of temperature of node 4. Then, considering 2 as internal node, we get difference eqn for node 2, from Eq. 8.69:

$$(T_{m,n})^{i+1} = Fo\cdot[(T_{m-1,n})^i + (T_{m+1,n})^i + (T_{m,n+1})^i + (T_{m,n-1})^i] + (1 - 4\cdot Fo)\cdot(T_{m,n})^i \quad \text{...(8.69)}$$

i.e.

$$T_2^{i+1} = (1 - 4\cdot Fo)\cdot T_2^i + Fo\cdot(T_4^i + T_4^i + T_1^i + 150) \quad \text{...(b)}$$

Eq. b is the explicit difference equation for temperature of node 2 at $(i + 1)$st time step.

For node 3:
This is corner node with convection. Elemental volume to be considered is 1/4 volume, a-3-d-b.

Applying energy balance, with all heat flow lines into the volume:

$$h\cdot\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)\cdot(T_a - T_3^i) + \left(k\cdot\frac{\Delta y}{2}\cdot\frac{T_1^i - T_3^i}{\Delta x}\right) + \left(k\cdot\frac{\Delta x}{2}\cdot\frac{T_4^i - T_3^i}{\Delta y}\right) = \rho\cdot\frac{\Delta x}{2}\cdot\frac{\Delta y}{2}\cdot C_p\cdot\frac{T_3^{i+1} - T_3^i}{\Delta\tau}$$

Dividing by k/4 and simplifying,

$$T_3^{i+1} = \left(1 - 4\cdot Fo - 4\cdot Fo\cdot\frac{h\cdot\Delta x}{k}\right)\cdot T_3^i + 2\cdot Fo\cdot\left(T_1^i + T_4^i + 2\cdot\frac{h\cdot\Delta x}{k}\cdot T_a\right) \quad \text{...(c)}$$

Eq. c is the explicit difference equation for temperature of node 3 at $(i + 1)$st time step.

For node 4:
This is an internal corner node with convection on two sides. Elemental volume to be considered is 3/4 volume, g-f-e-b-d-4.

Again, applying energy balance, with all heat flow lines into the volume:

$$\left[h\cdot\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)\cdot(T_a - T_4^i) + \left(k\cdot\frac{\Delta y}{2}\cdot\frac{T_5^i - T_4^i}{\Delta x}\right) + \left(k\cdot\Delta x\cdot\frac{150 - T_4^i}{\Delta y}\right) + \left(k\cdot\Delta y\cdot\frac{T_2^i - T_4^i}{\Delta x}\right) + \left(k\cdot\frac{\Delta x}{2}\cdot\frac{T_3^i - T_4^i}{\Delta y}\right)\right]$$

$$= \rho\cdot\frac{3\cdot\Delta x\cdot\Delta y}{4}\cdot C_p\cdot\frac{T_4^{i+1} - T_4^i}{\Delta\tau}$$

Dividing by 3k/4 and simplifying,

$$T_4^{i+1} = \left(1 - 4\cdot Fo - 4\cdot Fo\cdot\frac{h\cdot\Delta x}{3\cdot k}\right)\cdot T_4^i + \frac{Fo}{3}\cdot\left(4\cdot T_2^i + 4\cdot 150 + 2\cdot T_5^i + 2\cdot T_3^i + 4\cdot\frac{h\cdot\Delta x}{k}\cdot T_a\right) \quad \text{...(d)}$$

Eq. d is the explicit difference equation for temperature of node 4 at $(i + 1)$st time step.

For node 5:
This is a surface node with convection. Elemental volume to be considered is 1/2 volume, g-f-i-h-g.

Again, applying energy balance,

$$h \cdot \Delta x \cdot (T_a - T_5^i) + \left(k \cdot \frac{\Delta y}{2} \cdot \frac{T_6^i - T_5^i}{\Delta x}\right) + \left(k \cdot \Delta x \cdot \frac{150 - T_5^i}{\Delta y}\right) + \left(k \cdot \frac{\Delta y}{2} \cdot \frac{T_4^i - T_5^i}{\Delta x}\right) = \rho \cdot \Delta x \cdot \frac{\Delta y}{2} \cdot C_p \cdot \frac{T_5^{i+1} - T_5^i}{\Delta \tau}$$

Dividing by k/2 and simplifying,

$$T_5^{i+1} = \left(1 - 4 \cdot Fo - 2 \cdot Fo \cdot \frac{h \cdot \Delta x}{k}\right) \cdot T_5^i + Fo \cdot \left(T_4^i + T_6^i + 2 \cdot 150 + 2 \cdot \frac{h \cdot \Delta x}{k} \cdot T_a\right) \quad \text{...(e)}$$

Eq. e is the explicit difference equation for temperature of node 5 at $(i+1)$st time step.

For node 6:

This is identical to node 5. So, we get, by shifting node numbers by 1:

$$T_6^{i+1} = \left(1 - 4 \cdot Fo - 2 \cdot Fo \cdot \frac{h \cdot \Delta x}{k}\right) \cdot T_6^i + Fo \cdot \left(T_5^i + T_7^i + 2 \cdot 150 + 2 \cdot \frac{h \cdot \Delta x}{k} \cdot T_a\right) \quad \text{...(f)}$$

Eq. f is the explicit difference equation for temperature of node 6 at $(i+1)$st time step.

For node 7:

This is corner node with conduction from left, convection on the top, insulated on the right, and conduction from down. Elemental volume to be considered is 1/4 volume, k-7-p-j.

Writing the energy balance, remembering that right surface is insulated, and all heat flow lines into the volume, we get:

$$k \cdot \frac{\Delta y}{2} \cdot \frac{T_6^i - T_7^i}{\Delta x} + h \cdot \frac{\Delta x}{2} \cdot (T_a - T_7^i) + 0 + k \cdot \frac{\Delta x}{2} \cdot \frac{150 - T_7^i}{\Delta y} = \rho \cdot \frac{\Delta x}{2} \cdot \frac{\Delta y}{2} \cdot C_p \cdot \frac{T_7^{i+1} - T_7^i}{\Delta \tau}$$

Dividing by k/4 and simplifying,

$$T_7^{i+1} = \left(1 - 4 \cdot Fo - 2 \cdot Fo \cdot \frac{h \cdot \Delta x}{k}\right) \cdot T_7^i + 2 \cdot Fo \cdot \left(T_6^i + 150 + \frac{h \cdot \Delta x}{k} \cdot T_a\right) \quad \text{...(g)}$$

Eq. g is the explicit difference equation for temperature of node 7 at $(i+1)$st time step.

Now, we have to fix the upper limit of $\Delta\tau$ from stability criterion. To do that, we observe that in Eqs. a to f, the smallest coefficient of T_m^i is in Eq. c, i.e. $(1 - 4.\ Fo\ -4.Fo.h.\Delta x/k)$ must be positive. Putting this condition, we get:

$$1 - 4 \cdot Fo - 4 \cdot Fo \cdot \frac{h \cdot \Delta x}{k} \geq 0$$

i.e. $$Fo \leq \frac{1}{4 \cdot \left(1 + \frac{h \cdot \Delta x}{k}\right)}$$

i.e. $$\Delta\tau \leq \frac{(\Delta x)^2}{4 \cdot \alpha \cdot \left(1 + \frac{h \cdot \Delta x}{k}\right)}$$

i.e. $$\Delta\tau \leq 7.622 \text{ s}$$

This means that a time step less than 7.622 s has to be employed from stability criterion. Let us choose:

$$\Delta\tau := 5 \text{ s}$$

Then, $$Fo := \frac{\alpha \cdot \Delta\tau}{(\Delta x)^2} \quad \text{i.e. } Fo = 0.16$$

Substituting all relevant numerical values in Eq. a to f, we get the explicit difference equations as:

$$T_1^{i+1} = 0.352 \cdot T_1^i + 0.32 \cdot (T_3^i + T_2^i + 0.5) \quad \text{...(a)}$$

$$T_2^{i+1} = 0.36 \cdot T_2^i + 0.16 \cdot (T_4^i + T_4^i + T_1^i + 150) \quad \text{...(b)}$$

$$T_3^{i+1} = 0.344 \cdot T_3^i + 0.32 \cdot (T_1^i + T_4^i + 1) \quad \text{...(c)}$$

$$T_4^{i+1} = 0.35467 \cdot T_4^i + 0.05333 \cdot (4 \cdot T_2^i + 2 \cdot T_5^i + 2 \cdot T_3^i + 602) \quad \text{...(d)}$$

$$T_5^{i+1} = 0.352 \cdot T_5^i + 0.16 \cdot (T_4^i + T_6^i + 301) \quad \text{...(e)}$$

$$T_6^{i+1} = 0.352 \cdot T_6^i + 0.16 \cdot (T_5^i + T_7^i + 301) \quad \text{...(f)}$$

$$T_7^{i+1} = 0.352 \cdot T_7^i + 0.32 \cdot (T_6^i + 150.5) \quad \text{...(g)}$$

Initial temperature of the plate at $\tau = 0$ and $i = 0$, is given as 150°C.

i.e. $$T_1^0 = T_2^0 = T_3^0 = T_4^0 = T_5^0 = T_6^0 = T_7^0 = 150\text{°C}$$

Therefore, at the next time step $i = 1$, i.e. at $\Delta\tau = 5$ s, temperatures at nodes 1 to 7 can be explicitly calculated from Eqs. a to g. Then, calculate temperatures at the nodes for next time step of $\Delta\tau = 10$ s, using the same Eqs. a to g, since the temperatures at the previous time step are already calculated. Thus, march in time till we reach the time limit specified in the problem. Note that for 1 min. there are 12 time steps of 5 s each.

This calculation is easily done in Mathcad. We slightly change the notation for convenience in calculation: we write the superscripts as subscripts to work with matrix notation, as shown below.

In the small Mathcad program given below, LHS defines a function Temp(n) where n is the no. of time steps, which we can specify. Output is a vector containing step number total time elapsed, and node temperatures $T_1, T_2, ..., T_7$.

On the RHS, first 7 lines define the initial temperatures at the nodes, all equal to 150°C.

Then, a 'for loop' evaluates the finite difference Eqs. a to g, each 'new' node temperature being calculated in terms of the temperatures calculated in the previous time step. Here, the no. of time steps, 'n' can be changed since it is included in function definition on the LHS.

```
Temp(n) = | T1_o ← 150
          | T2_o ← 150
          | T3_o ← 150
          | T4_o ← 150
          | T5_o ← 150
          | T6_o ← 150
          | T7_o ← 150
          | for i ∈ 0, ..., n
          |   | T1_(i+1) ← 0.352·T1_i + 0.32·(T3_i + T2_i + 0.5)
          |   | T2_(i+1) ← 0.36·T2_i + 0.16·(T4_i + T4_i + T1_i + 150)
          |   | T3_(i+1) ← 0.344·T3_i + 0.32·(T1_i + T4_i + 1)
          |   | T4_(i+1) ← 0.35467·T4_i + 0.05333·(4·T2_i + 2·T5_i + 2·T3_i + 602)
          |   | T5_(i+1) ← 0.352·T5_i + 0.16·(T4_i + T6_i + 301)
          |   | T6_(i+1) ← 0.352·T6_i + 0.16·(T5_i + T7_i + 301)
          |   | T7_(i+1) ← 0.352·T7_i + 0.32·(T6_i + 150.5)
          | [i  5·i  T1_i  T2_i  T3_i  T4_i  T5_i  T6_i  T7_i]
```

Check: Temp(0) = [0 0 150 150 150 150 150 150 150] *(starting at time = 0)*

i = step no.; $\Delta\tau$ = one time step = 5 s; τ = time duration from beginning = $i.\ \Delta\tau$, s

	i	τ	T_1	T_2	T_3	T_4	T_5	T_6	T_7
Temp(12) =	[12	60	142.823	146.402	141.661	145.968	146.68	146.827	146.85]
Temp(24) =	[24	120	141.573	145.591	140.488	145.326	146.442	146.726	146.78]
Temp(60) =	[60	300	141.331	145.434	140.261	145.203	146.398	146.708	146.729]
Temp(120) =	[120	600	141.33	145.434	140.26	145.202	146.398	146.708	146.769]
Temp(180) =	[180	900	141.33	145.434	140.26	145.202	146.398	146.708	146.769]
Temp(240) =	[240	$1.2\cdot10^3$	141.33	145.434	140.26	145.202	146.398	146.708	146.769]

In the above Table, first column gives step number second column gives the time elapsed (seconds), and 5th column gives the temperature of node 3. We get:

Time (min)	Temperature of node 3 (deg.C)
0	150
1	141.661
2	140.488
5	140.261
10	140.26
15	140.26
20	140.26

i.e. steady state is reached after about 5 min. from start up.

8.9 Accuracy Considerations

As mentioned earlier, numerical methods yield approximate values as compared to 'exact analytical solutions'. This is due to the following errors inherent in numerical methods:

(i) Discretization error This is due to the approximation used in formulation of numerical solutions i.e. due to the error involved in writing the derivatives in terms of differences. Remember that we assume the temperature variation between adjacent nodes to be linear, which may not be so in practice. This is equivalent to considering only first two terms in the Taylor series expansion of the temperature at the given node. Generally, discretization error is cumulative; but if the function changes sign, it is possible that the errors may cancel. Discretization error is proportional to the square of the time step $\Delta\tau$ (or Δx). Therefore, smaller the mesh size, smaller the discretization error.

(ii) Round off error This is due to the fact that computer retains only 15 digits accuracy in a calculation (in double precision) and the rest of the digits are either chopped off or rounded off. When this is done continuously for a large number of calculations, error is carried over to successive calculations and the cumulative error can be significant. Obviously, the round off error is proportional to the total number of computations performed, and reduces as the mesh size increases.

Therefore, while aiming at reducing the error involved in numerical methods, we note that we have to deal with two opposing effects: if the mesh size Δx (or time step size $\Delta\tau$) is decreased, discretization error is reduced, but the round off error increases since the total number of calculations increases. So, practical way of approaching the solution is to start with a coarse mesh and then gradually refine the mesh size and observe if the results converge.

As a note of caution, it should be pointed out that getting an accurate solution of the nodal equations may not necessarily mean that an accurate solution to the physical problem has been obtained, if the formulation of nodal equations itself is erroneous. Therefore, as a check, some sort of energy balance using the final solution is also recommended.

8.10 Summary

While considering heat transfer in solids with complicated geometries and boundary conditions, and temperature dependent thermal properties, it is difficult to formulate 'exact' analytical solutions. In such cases, numerical methods are adopted to determine the temperature distribution and heat transfer rates.

In this chapter, we first considered the numerical solution of one-dimensional steady state conduction in cartesian, cylindrical and spherical coordinates. Then, two-dimensional conduction in cartesian coordinates was studied. Finally, numerical solutions for one-dimensional and two-dimensional transient conduction in cartesian coordinates was explained.

'Finite difference method' involves converting the partial differential equations of heat transfer into a set of coupled algebraic equations and then solving them. Analytical solution gives the temperature at any point in the medium; however, in numerical method, we divide the volume into discrete subvolumes and each subvolume is represented by a 'node' and the temperatures are determined at these discrete nodes.

Method adopted to convert the differential equations into a set of algebraic equations is to write an energy balance at each node. As a rule, all heat flow lines are considered to be flowing ***into*** the node considered. While writing the energy balance for a steady state problem, sum of all heat flows ***into*** the node must be equal to zero. Nodes at the boundaries for different boundary conditions are also handled in the same way, i.e. by writing energy balances at the boundary nodes. Care must be taken to see that at any boundary node, the volume considered must be the one appropriate to that node (i.e. half volume for a surface node, ¼ volume for an external corner node, ¾ volume for an internal corner node etc.)

Solution of the set of algebraic equations may be obtained by 'direct methods' or by 'iteration methods'. Direct methods are: Gaussian elimination method and Matrix inversion methods. Example of iteration method is the popular 'Gauss – Siedel iteration method' where one starts with the guess values of temperatures. These methods were explained briefly in this chapter.

While considering the numerical method for transient conduction, we again adopt the technique of writing the energy balances at the nodes; however, now we say that net energy flowing ***into*** a node results in a variation of the energy content of the subvolume represented by that node during the time interval $\Delta\tau$. In the 'explicit method', heat transfer and heat generation terms are considered at the 'previous time step' i, whereas in the 'implicit method', these terms are considered at the 'new time step', $i + 1$. In the explicit formulation, temperatures are obtained in a straightforward manner in terms of the values obtained at the previous time step. However, explicit method suffers from a disadvantage that we cannot use any time step we like and the solution becomes unstable unless the time step is below a particular value as dictated by 'stability criterion'. In implicit formulation, there is no such limitation on the time step i.e. any larger time step can be used resulting in smaller number of total calculations; however, at each step, all the equations have to be solved simultaneously.

While discussing the accuracy of the numerical solution, it was pointed out that smaller the mesh size, better the accuracy; but, now the total number of computations will be more and this introduces larger round off errors. Further, from a practical point of view, when convection boundary conditions are involved (which is invariably the case), the uncertainty in the value of heat transfer coefficients itself may be of the order of 20% and it is quite likely that the thermal properties may also be in error by 10 to 15%. Therefore, there is no point in having an unduly fine mesh. So, practical way of approaching the solution is to start with a coarse mesh and then progressively refine the mesh size, observing that the temperature values at given nodes go on converging.

Questions

1. When is a numerical solution adopted for a problem? What are its advantages and limitations?
2. Mention the methods used to convert partial differential equations of conduction heat transfer into finite difference equations.
3. Explain the energy balance procedure to obtain the finite difference formulation of one-dimensional conduction problem in cartesian coordinates.
4. Explain the energy balance procedure to obtain the finite difference formulation of one-dimensional conduction problem in cylindrical and spherical coordinates.
5. Explain the procedure of writing finite difference equation for an insulated boundary.
6. Explain the 'direct' and 'iterative' methods used for the solution of a system of algebraic equations.
7. 'Heat transfer problems involving variable thermal conductivity and radiation boundary conditions are difficult to handle' – explain this statement.
8. Give two examples of two-dimensional conduction where numerical methods are employed conveniently.
9. Finite difference formulation for a general interior node in a medium is given by:

$$T_{m-1,\,n} + T_{m+1,\,n} + T_{m,\,n+1} + T_{m,\,n-1} - 4 \cdot T_{m,\,n} + \frac{q_g \cdot (\Delta x)^2}{k} = \frac{T_m^{\,i+1} - T_m^{\,i}}{Fo}$$

   (i) Is the heat transfer in this medium steady or transient?
   (ii) Is there heat generation in the medium?
   (iii) Is the heat transfer one, two or three-dimensional?
   (iv) Is the nodal spacing constant or variable?
   (v) Is the thermal conductivity of the medium constant or variable?
10. Explain the method of handling an irregular boundary while writing finite difference equations.
11. How does the procedure of finite difference formulation for transient conduction differ from that for steady state conduction?
12. Explain the principle of getting 'explicit' and 'implicit' formulations for transient conduction.
13. Explain the 'stability criterion' when using explicit formulation for one-dimensional and two-dimensional transient conduction.
14. What are the relative advantages and disadvantages of explicit and implicit formulations?
15. Explain the types of errors inherent in numerical methods. How to reduce these errors?
16. How does the step size influence the discretization and round off errors?

Problems

One-dimensional steady state conduction:

1. A large plane wall of thickness $L = 0.5$ m, thermal conductivity $k = 14$ W/(mC), and surface area $A = 20$ m^2, has its left face maintained at a constant temperature of 150°C and the right face is exposed at ambient at 20°C, with a heat transfer coefficient of $h = 20$ W/(m^{2}C). Assuming a nodal spacing of 10 cm, and steady one-dimensional heat transfer, formulate the finite difference equations for all nodes and solve them to find the temperatures at all nodes. What is the rate of heat transfer through this wall?
2. A plane wall of thickness 0.1 m and $k = 20$ W/(mC) has uniform heat generation of 0.35 MW/m^3. It is insulated on one side and the other side is subjected to convection heat transfer with a fluid at 90°C flowing with a heat transfer coefficient of 550 W/(m^{2}C). Determine the temperature distribution in the wall by finite difference method.

3. A 30 mm diameter copper cable carries 200 A and has an electrical resistance of 5 milli-ohms per metre. Cable loses heat to the ambient air at 15°C with a convection coefficient of 20 W/(m²C). Determine the temperature distribution by numerical method. Compare the results with exact solution.
4. A 0.6 cm diameter solid aluminium sphere (k = 200 W/(mC) has an energy generation rate of 10^8 W/m³. Sphere loses heat from its outside surface to an ambient at 25°C by convection with a heat transfer coefficient of 150 W/(m²C). Calculate the steady state radial temperature distribution by the finite difference method by dividing the region into 6 elements, each of radial thickness Δr = 0.05 cm. Compare the results with exact solution.
5. Consider a straight fin of circular cross-section, 5 mm in diameter and 50 mm long (k = 14 W (mC)). Surface of the fin is exposed to ambient air at 20°C with a convection heat transfer coefficient of 80 W/(m²C). Base of the fin is maintained at 150°C. Assuming the tip of the fin to be insulated, determine the temperature distribution in the fin, heat transferred and the fin efficiency by finite difference method. Use 5 equal subdivisions along the length. Compare your results with the exact solution.
6. Consider an aluminium straight fin of square cross-section (4 mm × 4 mm), 2 cm long (k = 200 W (mC)). Surface of the fin is exposed to ambient air at 20°C with a convection heat transfer coefficient of 25 W/(m²C). Base of the fin is maintained at 150°C. Assuming that the tip of the fin is also losing heat by convection, determine the temperature distribution in the fin, heat transferred and the fin efficiency by finite difference method. Use 4 equal subdivisions along the length. Compare your results with the exact solution.

Two-dimensional steady state conduction:

7. A long rod of square cross-section (3 cm × 3 cm), has its top and bottom surfaces maintained at 0°C while the left and right surfaces are maintained at 50°C and 100°C respectively. Determine the steady state temperature distribution in the rod, using a node spacing of 1 cm.
8. A long rod of square cross-section (2 cm × 2 cm), has its top surface maintained at 120°C while each of the other three surfaces is maintained at 80°C. Determine the steady state temperature distribution in the rod, using a node spacing of 0.5 cm. Check the results with analytical solution.
9. Refer to the L-bar shown in Fig. Problem 8.9.
   If the thermal conductivity of the material is 15 W/(mC), find out the temperatures at all the nodes. (Note: Δx = Δy = 1 cm)

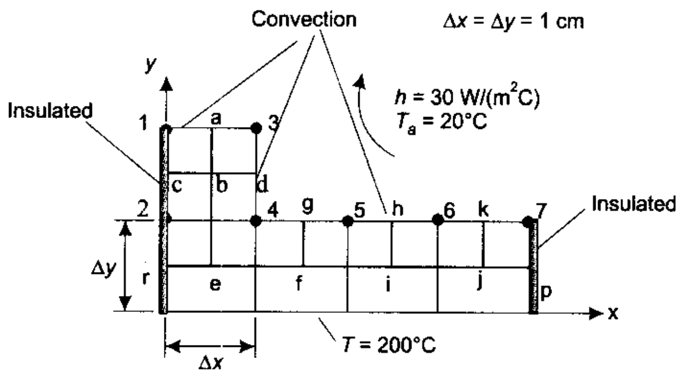


FIGURE Problem 8.9 Two-dimensional conduction in a L-bar

10. If in the L-bar shown in Fig. Problem 8.9 above, there is a heat generation at a rate of 1 MW/m³, all the other data remaining the same, determine the temperatures at all nodes.
11. If in the L-bar shown in Fig. Problem 8.9 above, if the right face is also subjected to convection conditions of the top surface, all the other data remaining the same, determine the temperatures at all nodes.
12. Consider a long bar of rectangular cross-section (6 cm wide × 9 cm height), with a thermal conductivity of 14 W/(mC). Top surface of the bar (with 60 mm width) is exposed to air at 90°C with a convection coefficient of 80 W/(m²C), while the other three surfaces are maintained at 35°C. Using a nodal spacing of 1.5 cm, determine the steady state temperature distribution in the bar and the heat transfer rate per unit length of the bar.
13. A gas duct made of fire brick, (k = 1 W/(mC)), has outer dimension of 4 m × 4 m. Gas passage area is 2 m × 2 m, centrally located. Inner walls are at a temperature of 900°C and the outer walls are at 40°C. Determine the temperature distribution in the wall and the heat transfer rate per metre length.

14. Consider two-dimensional steady state conduction in a region, 2 cm × 2 cm, with the boundary conditions as shown in Fig. Problem 8.14. For the material, k = 60 W/ (mC) and there is internal heat generation at a rate of 10^7 W/m^3. Using finite difference method, calculate the unknown node temperatures.

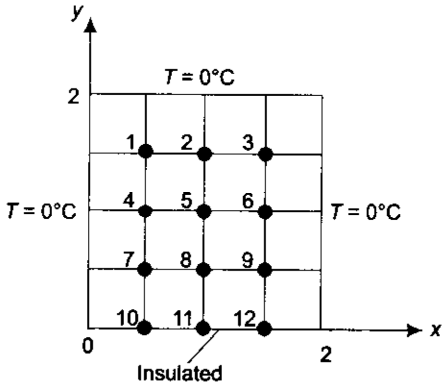


FIGURE Problem 8.14 Two-dimensional steady state conduction

One-dimensional transient conduction:

15. A very thick copper plate (k = 386 W/(mC), aa = 11 × 10^{-5} m^2/s) is initially at 400°C. Suddenly, its surface temperature is lowered to 20°C. Considering the plate as semi-infinite plate and using a mesh size Δx = 1 cm, calculate the temperature at x = 5 cm from the surface, 2 min. after lowering the surface temperature.
16. A water main is buried below the surface of soil which is initially at an uniform temperature of 25°C. Suddenly, the surface temperature drops to −30°C and is maintained so for a period of 60 days. Determine the depth at which the water mains must be placed to avoid freezing of water. Take properties of soil as: ρ = 2050 kg/m^3, k = 0.52 W/(mC), C_p = 1840 J/(kgK), α = 0.138 × 10^{-6} m^2/s. (Hint: Consider the soil as semi-infinite medium; calculate temperatures at distances upto 6 m below the surface and find the depth at which the temperature would be 0°C, by interpolation).
17. A 6 cm thick steel plate (α = 1.6 × 10^{-5} m^2/s, k = 60 W/(mC)), is initially at an uniform temperature of 250°C. It is suddenly exposed to a cold air stream at 20°C on both the surfaces, with a heat transfer coefficient of 350 W/(m^{2}C). Determine the centre plane temperature at τ = 5, 10 and 15 min. from starting of cooling. Use explicit formulation with a mesh size of Δx = 1 cm.
18. Two ends of a steel rod 1.2 cm diameter and 2.5 m long, are maintained at 250°C and 50°C and the curved surface of the rod is perfectly insulated. Suddenly, an electric current is passed through the rod, causing heat generation in the rod at an uniform rate of 3000 W/m^3. Find the temperature distribution in the rod for the first five time increments. Take k = 35 W/(mC) and α = 1.5 × 10^{-5} m^2/s.

Two-dimensional transient conduction:

19. The L-bar shown in Fig. Problem 8.9 is initially at an uniform temperature of 200°C. Its top surface is suddenly exposed to convection with an air stream at 20°C with a convection coefficient of 80 W/(m^{2}C). Bottom surface is maintained at 200°C throughout and the left and right surfaces are insulated as shown. Taking k = 15 W/(mC) and α = 3.2 × 10^{-6} m^2/s, calculate the temperature of node 3 after 1, 3, 5, 10 and 30 min. Use explicit formulation.
20. A steel bar of 3 cm × 3 cm cross-section is initially at an uniform temperature of 500°C. (α = 1.0 × 10^{-5} m^2/s, k = 35 W/(mC)). Suddenly, all the 4 surfaces of the bar are exposed to an air stream at 20°C with a heat transfer coefficient of 120 W/(m^{2}C). Using explicit formulation and a mesh size of Δx = Δy = 0.5cm, calculate the centre temperature at τ = 1, 5 and 10 min. after the start of cooling. (Hint: Use symmetry consideration—consider only a quarter of the cross-section).

CHAPTER

9

Forced Convection

9.1 Introduction

In the previous chapters, we studied about conduction heat transfer, where heat transfer was a molecular phenomenon and was considered mainly in solids; convection was mentioned only in passing and was considered only as a boundary condition while analysing conduction heat transfer.

In convection heat transfer, there is a flow of fluid associated with heat transfer and the energy transfer is mainly due to bulk motion of the fluid. When the flow of fluid is caused by an external agency such as a fan or pump or due to atmospheric disturbances, the resulting heat transfer is known as 'Forced convection heat transfer'; when the flow of fluid is due to density differences caused by temperature differences, the heat transfer is said to be by 'Natural (or free) convection'. For example, if air is blown on a hot plate by a blower, heat transfer occurs by forced convection, whereas, a hot plate simply hung in air will lose heat by natural convection.

In this chapter, we shall study about forced convection heat transfer. Since there is a flow of fluid involved in convection heat transfer, it is clear that the flow field will influence the heat transfer greatly. Mathematical solution of convection heat transfer will, therefore, require the simultaneous solution of differential equations resulting by the application of conservation of mass, conservation of momentum and conservation of energy, under the constraints of given boundary conditions. For a three-dimensional fluid flow, mathematical solution of the resulting differential equations is a formidable task and it is usual to make many simplifying assumptions to get a mathematical solution. Still, it must be stated that exact mathematical solutions, even for simple convection heat transfer cases, are rather complicated and it is common practice to resort to empirical relations for solutions of problems involving convection heat transfer. These empirical relations are obtained by researchers after performing large number of experiments for several practically important situations and are presented in terms of non-dimensional numbers.

In this chapter, we shall first describe the physical mechanism of forced convection and then mention about the convective heat transfer coefficient and various factors affecting the same. Then, we shall discuss about velocity and thermal boundary layers. Application of conservation of mass, momentum and energy in respect of the boundary layer will be demonstrated next. We shall not rigorously solve these equations, but will only mention the methods of solution, since our emphasis will be on practical solutions with the use of empirical relations. Then, we present several empirical relations to determine friction and heat transfer coefficients for flow over different geometries such as a flat plate, cylinder and sphere for flow under laminar and turbulent conditions. Finally, flow inside tubes will be considered and determination of heat transfer coefficient by analogy with the mechanism of fluid flow will be explained.

9.2 Physical Mechanism of Forced Convection

Consider a hot iron block whose surface is at a temperature T_s. Let this surface be cooled by a fluid at a temperature T_a, flowing over its surface at a velocity U, as shown in Fig. 9.1.

We know that heat will be carried away from the hot iron block by the flowing fluid and the block will cool. We also know that if the velocity of the fluid is increased, more heat is carried away and the block will be cooled faster. For the purpose of analysis, we quantify the preceding statement by a dimensionless number called,